



TITLE:

An algebraic study of extension algebras

AUTHOR(S):

Kato, Syu

CITATION:

Kato, Syu. An algebraic study of extension algebras. American Journal of Mathematics 2017, 139(3): 567-615

ISSUE DATE:

2017-06

URL:

<http://hdl.handle.net/2433/230135>

RIGHT:

This article appeared in the American Journal of Mathematics, Volume 139, Issue 3, 2017, pages 576-615, Copyright © 2017, Johns Hopkins University Press.; This is not the published version. Please cite only the published version.; この論文は出版社版でありません。引用の際には出版社版をご確認ご利用ください。

An algebraic study of extension algebras

Syu KATO*

October 20, 2016

Abstract

We present simple conditions which guarantee a geometric extension algebra to behave like a variant of quasi-hereditary algebras. In particular, standard modules of affine Hecke algebras of type BC, and the quiver Schur algebras are shown to satisfy the Brauer-Humphreys type reciprocity and the semi-orthogonality property. In addition, we present a new criterion of purity of weights in the geometric side. This yields a proof of Shoji's conjecture on limit symbols of type B [Shoji, Adv. Stud. Pure Math. 40 (2004)], and the purity of the exotic Springer fibers [K, Duke Math. 148 (2009)]. Using this, we describe the leading terms of the C^∞ -realization of a solution of the Lieb-McGuire system in the appendix. In [K, Duke Math. 163 (2014)], we apply the results of this paper to the KLR algebras of type ADE to establish Kashwara's problem and Lusztig's conjecture.

Introduction

In representation theory of an algebra associated to a root datum, study of a geometric extension algebra plays a major rôle. Introduced by Ginzburg [Gin85] in his study of affine Hecke algebras, it appeared in the study of affine Hecke algebras [Lus88, Ari96, CG97, K09, VV11b], the BGG categories [Soe90, BGS96, ABG, Sch11a], the Springer correspondence [CG97, Ach11, K15, Rid12], quantum loop algebras [Nak06], the Khovanov-Lauda-Rouquier algebras [Zhe08, Web10, VV11a], the quiver Schur algebras [VV99, SW11], and so on. Once appeared, it produces deep results in the spirit of the Kazhdan-Lusztig conjecture [BB81, BK81] and the Koszul-Langlands duality [BGS96, Soe00, ABG].

However, not much is known about the standard modules of these geometric extension algebras. From the viewpoint of highest weight category [CPS88], we have several natural expectations on standard modules. For example, they should be indecomposable with simple heads, and filter projective modules. These are not a part of the general theory of geometric extension algebras [CG97] §8, and even its indecomposability is usually guaranteed by rather ad-hoc induction arguments.

Actually, several criteria which guarantee nice behavior of geometric extension algebras are known [MV87, BGS96]. The problem is that many of the geometric extension algebras arising from representation theory fail to satisfy

*Department of Mathematics, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502, Japan. E-mail: syuchan@math.kyoto-u.ac.jp

such criteria. In particular, the resulting geometric extension algebras rarely give rise to highest weight categories.

The goal of the present paper is to supply some geometric conditions (\clubsuit) which cover some algebras that are not covered by previous results, and to deduce their representation-theoretic consequences. Such an analysis, together with its algebraic interpretations, yields a proof of Shoji's conjecture in this paper. In addition, it serves as a basis of our proofs of Kashiwara's problem and Lusztig's conjecture in [K14].

Let G be a connected algebraic group acting on a variety \mathfrak{X} over \mathbb{C} with finitely many orbits $\{\mathbb{O}_\lambda\}_{\lambda \in \Lambda}$ labeled by Λ . Let IC_λ be the minimal extension of the constant sheaf on \mathbb{O}_λ (see e.g. [BBD82]). We assume the following three conditions unless stated otherwise (here we employ an extra condition $(\spadesuit)'$ in order to simplify the statements in this introduction):

- $(\spadesuit)'$ For each $\lambda \in \Lambda$, the G -orbit \mathbb{O}_λ has a connected stabilizer G_λ ;
- $(\clubsuit)_1$ The algebra $A_{(G, \mathfrak{X})}$ defined in the below is pure of weight zero;
- $(\clubsuit)_2$ The sheaf IC_λ is pointwise pure (of some weight) for every $\lambda \in \Lambda$.

Here we count weights by considering its positive characteristic analogue. For each $\lambda \in \Lambda$, we have a natural morphism $\psi_\lambda : H_G^\bullet(\mathrm{pt}) \rightarrow H_G^\bullet(\mathbb{O}_\lambda)$ (between equivariant cohomologies; see e.g. [BL94]) of graded algebras. The condition $(\clubsuit)_2$ is usually rather difficult to verify. For this, our analysis shows:

Theorem A (= Theorem 1.6). *We assume $(\clubsuit)_1$, but not $(\clubsuit)_2$. We have $(\clubsuit)_2$ as a consequence of the following two conditions:*

1. *For each $\lambda, \mu \in \Lambda$, the stalk of IC_λ along \mathbb{O}_μ satisfies the parity vanishing;*
2. *We have $\ker \psi_\lambda \not\subset \ker \psi_\mu$ for every $\mu \prec \lambda \in \Lambda$.*

The geometric extension algebra associated to such a pair (G, \mathfrak{X}) is:

$$A = A_{(G, \mathfrak{X})} := \bigoplus_{\lambda, \mu} \mathrm{Ext}_{D_G^b(\mathfrak{X})}^\bullet(L_\lambda \boxtimes \mathrm{IC}_\lambda[\dim \mathbb{O}_\lambda], L_\mu \boxtimes \mathrm{IC}_\mu[\dim \mathbb{O}_\mu]),$$

where L_λ is a self-dual non-zero graded vector space for each $\lambda \in \Lambda$. This makes $\{L_\lambda\}_\lambda$ a complete collection of self-dual simple graded modules of A . Geometric extension algebras arising from affine Hecke algebras of type A [CG97], type BC [K09], the KLR algebras of type ADE [VV11a, K14], the quiver Schur algebras [Lu90a, VV99], and the algebra which governs the BGG category [Soe90, BGS96] satisfy our condition. In addition, our condition is stable under the restriction to a locally closed subset (cf. Corollary 4.4 and Lemma B.2). Hence, one obtains many intermediate varieties and algebras (with some representation-theoretic significance) that are connected to each other (see [KP82] for example).

Theorem B (= Theorem 3.5). *The global dimension of A is finite.*

This is new for the quiver Schur algebras and the KLR algebras. For more precise explanation on the latter, see [K14] (cf. [BKM12]). For affine Hecke algebras, this is standard by taking filtered quotients (cf. [NO82] D.VII). In this case, an explicit bi-resolution of A is constructed by Opdam-Solleveld [OS09].

Since we deal with a variety \mathfrak{X} , we have a closure ordering among G -orbits that we denote by \prec . Let $A\text{-gmod}$ be the category of finitely generated graded A -modules. Let P_λ be the projective cover of L_λ . For $M \in A\text{-gmod}$ and $j \in \mathbb{Z}$, we denote by $M \langle j \rangle$ the grade j shift of M . We define

$$\tilde{K}_\lambda := P_\lambda / \left(\sum_{\mu \prec \lambda, f \in \text{hom}_A(P_\mu, P_\lambda)} \text{Im} f \right) \quad \text{and} \quad K_\lambda := \tilde{K}_\lambda / \left(\sum_{f \in \text{hom}_A(P_\lambda, \tilde{K}_\lambda) > 0} \text{Im} f \right),$$

where $\text{hom}_A(M, N)$ is the direct sum of the space of degree j homomorphisms $\text{hom}_A(M, N)^j := \text{Hom}_{A\text{-gmod}}(M \langle j \rangle, N)$ (for each $M, N \in A\text{-gmod}$).

For $M \in A\text{-gmod}$, we define its graded character as:

$$\text{gch } M := \sum_{\lambda \in \Lambda, j \in \mathbb{Z}} t^j [M : L_\lambda \langle j \rangle]_0 [\lambda] \in \bigoplus_{\lambda \in \Lambda} \mathbb{Z}((t))[\lambda],$$

where $[M : L_\lambda \langle j \rangle]_0 \in \mathbb{Z}_{\geq 0}$ is the multiplicity of $L_\lambda \langle j \rangle$ in M . We define

$$[M : L_\lambda] := \sum_{j \in \mathbb{Z}} [M : L_\lambda \langle j \rangle]_0 t^j \in \mathbb{Z}((t))$$

and define $\{[M : \tilde{K}_\lambda]\}_\lambda$ by

$$\text{gch } M = \sum_{\lambda \in \Lambda} [M : \tilde{K}_\lambda] \text{gch } \tilde{K}_\lambda.$$

Theorem C (= Theorems 1.8, 3.9, and 4.1 + Corollary 3.12). *Under the above setting, we have:*

1. Each \tilde{K}_λ admits a separable decreasing filtration whose associated graded is a direct sum of grading shifts of K_λ . In addition, we have

$$[\tilde{K}_\lambda : L_\lambda] = \sum_{i \geq 0} t^i \dim H_{\text{Stab}_G(x_\lambda)}^i(\text{pt});$$

2. For each $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned} \text{ext}_A^\bullet(\tilde{K}_\lambda, \tilde{K}_\mu) &= \{0\} \quad \text{if } \lambda \not\preceq \mu \quad \text{and} \\ \text{ext}_A^i(\tilde{K}_\lambda, K_\mu^*) &= \begin{cases} \mathbb{C} & (\lambda = \mu, i = 0) \\ \{0\} & (\text{otherwise}) \end{cases}, \end{aligned}$$

where K_μ^* is the graded dual of K_μ regarded as an A -module;

3. We have

$$[P_\lambda : \tilde{K}_\mu] = [K_\mu : L_\lambda] \quad \text{for every } \lambda, \mu \in \Lambda;$$

4. Each P_λ admits a finite filtration whose associated graded is a direct sum of grading shifts of $\{\tilde{K}_\mu\}_\mu$.

Note that the non-zero map in Theorem C 2) has its image L_λ since essentially the LHS is concentrated in degree ≥ 0 , and the RHS is concentrated in degree ≤ 0 . Also, Theorem C 1) yields the Cartan determinant formula of A (Corollary 3.13) and a variant of the Lusztig-Shoji algorithm (Remark 3.14 2)).

For the proof of Theorem C, we make a link between the mixed geometry of \mathfrak{X} and the homological algebra of A in §2, which might be of independent interest. One important consequence there is a description of the minimal projective resolution of \tilde{K}_λ in terms of the $!$ -extension of a constant sheaf on \mathbb{O}_λ (Corollary 2.10).

Example D. Let $G = PGL(3, \mathbb{C})$ and let $\mathfrak{X} = \mathcal{N}$ be the nilpotent cone of \mathfrak{sl}_3 . Let \mathfrak{t} be the Cartan subalgebra of \mathfrak{sl}_3 . We have $A = \mathbb{C}\mathfrak{S}_3 \ltimes \mathbb{C}[\mathfrak{t}]$ with $A^0 = \mathbb{C}\mathfrak{S}_3$ and $\deg \mathfrak{t}^* = 2$. We have $\Lambda = \{\text{triv} \succ \text{ref} \succ \text{sgn}\}$ and

$$\text{gch } K_{\text{triv}} = [\text{triv}], \text{gch } K_{\text{ref}} = [\text{ref}] + t^2[\text{triv}], \text{gch } K_{\text{sgn}} = [\text{sgn}] + (t^2 + t^4)[\text{ref}] + t^6[\text{triv}].$$

Then, the matrix $[P : L] := ([P_\lambda : L_\gamma])_{\lambda, \gamma}$ is presented as:

$$[P : L] = [P : \tilde{K}][\tilde{K} : K][K : L] = {}^t[K : L][\tilde{K} : K][K : L],$$

where $[\tilde{K} : K]$ is the graded character transition matrix. This reads as:

$$\begin{aligned} & \frac{1}{(1-t^4)(1-t^6)} \begin{pmatrix} 1 & t^2+t^4 & t^6 \\ t^2+t^4 & 1+t^2+t^4+t^6 & t^2+t^4 \\ t^6 & t^2+t^4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t^2 & t^6 \\ 0 & 1 & t^2+t^4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-t^2} & 0 \\ 0 & 0 & \frac{1}{(1-t^4)(1-t^6)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t^2 & 1 & 0 \\ t^6 & t^2+t^4 & 1 \end{pmatrix}. \end{aligned}$$

In [Sho04] 3.13, Shoji conjectured that his limit Green function of type B gives the graded character of a coinvariant ring of a Weyl group of type B. Combining the results of Achar-Henderson [AH08] with Theorem C and our previous results, we prove:

Theorem E (Shoji's conjecture for type B = Corollaries 5.7, 5.8, and 5.3). *Let $G = Sp(2n, \mathbb{C})$ and let $\mathfrak{X} = \mathfrak{N}$ be its exotic nilpotent cone [K09]. Then, the modules K_λ arising from A are coinvariant algebras of type B. In particular, their graded characters are calculated by the Lusztig-Shoji algorithm. Moreover, every exotic Springer fiber has a pure homology.*

We note that Shoji-Sorlin [SS12] independently proved the last part of Theorem E by a completely different method. Also, Theorem E implies [AH08] Conjecture 6.4.

As a bonus of Theorem E, we prove that the graded modules appearing from the Lieb-McGuire integrable systems [HO97] are exactly Shoji's coinvariant algebras in Appendix A.

The organization of this paper is as follows: In the first section, we formulate two conditions (\spadesuit) and (\clubsuit) and our main results (Theorem A and the main part of Theorem C). In the second section, we translate an iteration of distinguished triangles to a complex of projective modules to prove the main part of Theorem C. In the third section, we prove Theorem B and numerical consequences of Theorem C including the Brauer-Humphreys type reciprocity and the Cartan determinant formula. In the fourth section, we complete the proof of Theorem C, and show that our conditions are invariant under restrictions. Finally, we prove Shoji's conjecture in the fifth section. The appendices are devoted to the analysis of the Lieb-McGuire system and a proof of Theorem A.

Theorem C 3) shows that we have two versions of standard modules and they are deeply incorporated into the formulation. In addition, two versions of

standard modules actually differ in Example D. Therefore, our result mainly points different direction from that of Cline-Parshall-Scott [CPS88]. Also, there is a notion of affine cellularity due to König-Xi [KX12], which provides a framework for algebraic results similar to ours. However, their algebraic conditions seem rather difficult to verify compared with our geometric conditions. (For example, the combination of [K09] and §1 yields homological/categorical consequences parallel to the theory of affine cellular algebras for the affine Hecke algebras of type BC with arbitrary rank and arbitrary real parameters while the affine cellularity of the affine Hecke algebras of type BC is known only in rank two case with generic real parameters [GM11].) In addition, our approach gives some more precise information on A than affine cellularity (in a sense) as our approach naturally gives a variant of the Kazhdan-Lusztig polynomials given geometrically.

Acknowledgement: This is a revised and expanded version of the first part of the paper “PBW bases and KLR algebras” arXiv:1203.5254. We divided it into two pieces since we learned that this part were invisible in the previous version of the above mentioned paper. The author is indebted to Masaki Kashiwara for helpful discussions and comments. He is also indebted to Yoshiyuki Kimura for helpful comments and pointing out some errors. He is also grateful to Ryosuke Kodera for his question which leads him to Theorem 4.1, Eric Opdam for suggesting him to study the material presented in Appendix A, Julia Sauter for giving me some comments, and Wolfgang Soergel for suggesting him to study extension algebras. This research was supported in part by JSPS Grant-in-Aid for Young Scientists (B) 23-740014, JSPS Grant-in-Aid for Scientific Research (B) 26287004, and Kyoto University Jung-Mung Program.

Convention

An algebra R is a (not necessarily commutative) unital \mathbb{C} -algebra. A variety \mathfrak{X} is a separated reduced scheme \mathfrak{X}_0 of finite type over some localization \mathbb{Z}_S of \mathbb{Z} specialized to an algebraically closed field \mathbb{k} that we specify. It is called a G -variety if we have an action of a connected affine algebraic group scheme G flat over \mathbb{Z}_S on \mathfrak{X}_0 specialized to \mathbb{k} .

We fix some prime ℓ and fix an *identification* $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ once for all. Let us denote by $D^b(\mathfrak{X})$ (resp. $D^+(\mathfrak{X})$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on \mathfrak{X} , and denote by $D_G^+(\mathfrak{X})$ the G -equivariant derived category of \mathfrak{X} . We have a natural forgetful functor $D_G^+(\mathfrak{X}) \rightarrow D^+(\mathfrak{X})$, whose preimage of $D^b(\mathfrak{X})$ is denoted by $D_G^b(\mathfrak{X})$. For an object of $D_G^b(\mathfrak{X})$, we may denote its image in $D^b(\mathfrak{X})$ by the same letter.

Let \mathbf{vec} be the category of \mathbb{Z} -graded vector spaces (over \mathbb{C}) bounded from the below so that its objects have finite-dimensional graded pieces. In particular, for $V = \oplus_{i \gg -\infty} V^i \in \mathbf{vec}$, its graded dimension $\mathbf{gdim} V := \sum_i t^i \dim V^i \in \mathbb{Z}((t))$ makes sense (with t being indeterminate). We define $V\langle m \rangle$ by setting $(V\langle m \rangle)^i := V^{i-m}$.

In this paper, a graded algebra A is always a \mathbb{C} -algebra whose underlying space is in \mathbf{vec} . Let $A\text{-gmod}$ be the category of finitely generated graded A -modules. For $E, F \in A\text{-gmod}$, we define $\mathrm{hom}_A(E, F)$ to be the direct sum of graded A -module homomorphisms $\mathrm{hom}_A(E, F)^j$ of degree j ($= \mathrm{Hom}_{A\text{-gmod}}(E\langle j \rangle, F)$). We employ the same notation for extensions (i.e. $\mathrm{ext}_A^i(E, F) = \bigoplus_{j \in \mathbb{Z}} \mathrm{ext}_A^i(E, F)^j$).

We denote by $\text{lrr } A$ the set of isomorphism classes of graded simple modules of A , and denote by $\text{lrr}_0 A$ the set of isomorphism classes of graded simple modules of A up to grading shifts. Two graded algebras are said to be (graded) Morita equivalent if their (graded) module categories are equivalent. For a graded A -module E , we denote its head by $\text{hd } E$, and its socle by $\text{soc } E$.

For $Q(t) \in \mathbb{Q}(t)$, we set $\overline{Q(t)} := Q(t^{-1})$. We denote the set of isomorphism classes of representations of a finite group H by $\text{lrr } H$. For each $\chi \in \text{lrr } H$, we denote its dual representation by χ^\vee . For derived functors $\mathbb{R}F$ or $\mathbb{L}F$ of some functor F , we represent its arbitrary graded piece (of its homology complex) by \mathbb{R}^*F or \mathbb{L}^*F , and the direct sum of whole graded pieces by $\mathbb{R}^\bullet F$ or $\mathbb{L}^\bullet F$. For example, $\mathbb{R}^*F \cong \mathbb{R}^*G$ means that $\mathbb{R}^i F \cong \mathbb{R}^i G$ for every $i \in \mathbb{Z}$, while $\mathbb{R}^\bullet F \cong \mathbb{R}^\bullet G$ means that $\bigoplus_i \mathbb{R}^i F \cong \bigoplus_i \mathbb{R}^i G$.

When working on some sort of derived category, we suppress \mathbb{R} or \mathbb{L} , or the category from the notation for the sake of simplicity in case there is only small risk of confusion.

1 The conditions $(\spadesuit), (\clubsuit)$ and a main result

For a while, we fix our ground field \mathbb{k} to be either \mathbb{C} or the algebraic closure of a finite field. Let G be a connected (affine) algebraic group. Let \mathfrak{X} be a G -variety. Let $\underline{\Lambda}$ be the labeling set of G -orbits of \mathfrak{X} . We denote by $\mathbb{O}_{\underline{\lambda}}$ the G -orbit corresponding to $\underline{\lambda} \in \underline{\Lambda}$. Let $C_{\underline{\lambda}}$ be the component group of the stabilizer $G_{\underline{\lambda}}$ of a closed point of $\mathbb{O}_{\underline{\lambda}}$. It is always a finite group. Let Λ be the set of conjugacy classes of pairs $(\underline{\lambda}, \xi)$ with $\underline{\lambda} \in \underline{\Lambda}$ and $\xi \in \text{lrr } C_{\underline{\lambda}}$. For $\lambda = (\underline{\lambda}, \xi) \in \Lambda$, we set $\mathbb{O}_\lambda := \mathbb{O}_{\underline{\lambda}}$, $C_\lambda := C_{\underline{\lambda}}$, $G_\lambda := G_{\underline{\lambda}}$, and $\lambda^\vee := (\underline{\lambda}, \xi^\vee)$. For $\lambda, \mu \in \Lambda$, we write $\lambda \preceq \mu$ if $\mathbb{O}_\lambda \subset \overline{\mathbb{O}_\mu}$, and write $\lambda \sim \mu$ if $\mathbb{O}_\lambda = \mathbb{O}_\mu$. We assume the following condition (\spadesuit) in the below:

Condition 1.1 (Condition (\spadesuit)). The set $\underline{\Lambda}$ (or equivalently Λ) is finite. For each $\lambda \in \Lambda$, we fix a \mathbb{k} -valued point $x_\lambda \in \mathbb{O}_\lambda$ such that $x_\lambda = x_\mu$ if $\lambda \sim \mu$.

We have a (relative) dualizing complex $\omega_{\mathfrak{X}} := p^! \mathbb{C} \in D_G^b(\mathfrak{X})$, where $p : \mathfrak{X} \rightarrow \text{pt}$ is the G -equivariant structure map. We have a dualizing functor

$$\mathbb{D} : D_G^b(\mathfrak{X})^{\text{op}} \ni C^\bullet \mapsto \text{Hom}^\bullet(C^\bullet, \omega_{\mathfrak{X}}) \in D_G^b(\mathfrak{X}).$$

We have a \mathbb{D} -autodual t -structure of $D_G^b(\mathfrak{X})$ whose truncation functor and perverse cohomology functor are denoted by τ and ${}^p H$, respectively. In particular, \mathbb{D} induces an equivalence of categories $\tau_{\geq 0} D_G^b(\mathfrak{X})^{\text{op}} \cong \tau_{\leq 0} D_G^b(\mathfrak{X})$ and each $\mathcal{E} \in D_G^b(\mathfrak{X})$ admits a distinguished triangle

$$\tau_{< m} \mathcal{E} \rightarrow \mathcal{E} \rightarrow \tau_{\geq m} \mathcal{E} \xrightarrow{+1} \tau_{< m} \mathcal{E}[1]$$

for every $m \in \mathbb{Z}$.

For each $\lambda := (\underline{\lambda}, \xi) \in \Lambda$, we have a G -equivariant irreducible local system $\underline{\xi}$ on \mathbb{O}_λ induced from ξ . We have inclusions $i_\lambda : \{x_\lambda\} \hookrightarrow \mathfrak{X}$ and $j_\lambda : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$. Let

$$\mathbb{C}_\lambda := (j_\lambda)_! \underline{\xi}[\dim \mathbb{O}_\lambda] \quad \text{and} \quad \mathbb{IC}_\lambda := (j_\lambda)_! \underline{\xi}[\dim \mathbb{O}_\lambda] \quad (1.1)$$

be the extension by zero and the minimal extension, which we regard as objects

of $D_G^b(\mathfrak{X})$. We denote by

$$\begin{aligned}\mathrm{Ext}_G^\bullet(\bullet, \bullet) &: D_G^b(\mathfrak{X})^{op} \times D_G^b(\mathfrak{X}) \longrightarrow D^+(\mathrm{pt}) \\ \mathrm{Ext}^\bullet(\bullet, \bullet) &: D^b(\mathfrak{X})^{op} \times D^b(\mathfrak{X}) \longrightarrow D^b(\mathrm{pt})\end{aligned}$$

the Ext (as bifunctors) of $D_G^b(\mathfrak{X})$ and $D^b(\mathfrak{X})$, respectively.

For each $\lambda \in \Lambda$, we fix $L_\lambda \in D^b(\mathrm{pt})$ as a non-zero graded vector space with a trivial differential which satisfies the duality condition $L_{\lambda^\vee} \cong L_\lambda^*$. We set

$$\mathcal{L} := \bigoplus_{\lambda \in \Lambda} L_\lambda \boxtimes \mathrm{IC}_\lambda \in D_G^b(\mathfrak{X}).$$

By construction, we find an isomorphism $\mathcal{L} \cong \mathbb{D}\mathcal{L}$.

We form a graded Yoneda algebra

$$A_{(G, \mathfrak{X})} = \bigoplus_{i \in \mathbb{Z}} A_{(G, \mathfrak{X})}^i := \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_G^i(\mathcal{L}, \mathcal{L})$$

whose degree is the cohomological degree. We denote by $B_{(G, \mathfrak{X})}$ the algebra $A_{(G, \mathfrak{X})}$ obtained by taking $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathrm{IC}_\lambda$ (and call it the basic ring of $A_{(G, \mathfrak{X})}$). The algebra $B_{(G, \mathfrak{X})}$ is graded Morita equivalent to $A_{(G, \mathfrak{X})}$ (see Lemma 1.11 in the below), and hence all the arguments in the below are independent of the choice of \mathcal{L} , that we suppress for simplicity. We also drop (G, \mathfrak{X}) in case the meaning is clear from the context. It is standard that $B_{(G, \mathfrak{X})}$ is non-negatively graded (as $\mathrm{Ext}_G^{<0}(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \equiv \{0\}$) and $\{L_\lambda\}_{\lambda \in \Lambda}$ forms a complete collection of graded simple A -modules up to grading shifts (as $\bigoplus_{\lambda \in \Lambda} \mathrm{end}_{\mathbb{C}}(L_\lambda)$ is the maximal graded semisimple quotient of A by $\dim \mathrm{Ext}_G^0(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = \delta_{\lambda, \mu}$). We have $\mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L}) \in D^b(\mathfrak{X})$, which implies $\dim \mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L}) < \infty$. By the Serre spectral sequence

$$H_G^\bullet(\mathrm{pt}) \otimes_{\mathbb{C}} \mathrm{Ext}^\bullet(\mathcal{L}, \mathcal{L}) \Rightarrow \mathrm{Ext}_G^\bullet(\mathcal{L}, \mathcal{L}) \cong A, \quad (1.2)$$

we conclude that $A \in \mathrm{vec}$.

Lemma 1.2 (cf. Joshua [Jos97] §6). *We have an isomorphism $A \cong A^{op}$ of graded algebras induced by an isomorphism $\mathcal{L} \cong \mathbb{D}\mathcal{L}$. In addition, the graded dual of L_λ is naturally isomorphic to L_{λ^\vee} as a graded A -module for each $\lambda \in \Lambda$.*

Proof. The isomorphism $\mathbb{D}\mathcal{L} \cong \mathcal{L}$ induces an isomorphism $\mathbb{D}(L_\lambda \boxtimes \mathrm{IC}_\lambda) \cong L_{\lambda^\vee} \boxtimes \mathrm{IC}_{\lambda^\vee}$ for each $\lambda \in \Lambda$. This determines the both of $L_\lambda^* \cong L_{\lambda^\vee}$ and $\mathbb{D}\mathrm{IC}_\lambda \cong \mathrm{IC}_{\lambda^\vee}$ by the simplicity of IC_λ (up to a scalar).

For each $\lambda, \mu \in \Lambda$, we have a natural identification

$$\mathrm{hom}_{\mathbb{C}}(L_\lambda, L_\mu) \boxtimes \mathrm{Ext}_G^\bullet(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) = \mathrm{Ext}_G^\bullet(L_\lambda \boxtimes \mathrm{IC}_\lambda, L_\mu \boxtimes \mathrm{IC}_\mu) \subset A.$$

For each $\lambda, \mu, \gamma \in \Lambda$, this identification factors the multiplication map

$$\mathrm{Ext}_G^\bullet(L_\lambda \boxtimes \mathrm{IC}_\lambda, L_\mu \boxtimes \mathrm{IC}_\mu) \times \mathrm{Ext}_G^\bullet(L_\mu \boxtimes \mathrm{IC}_\mu, L_\gamma \boxtimes \mathrm{IC}_\gamma) \rightarrow \mathrm{Ext}_G^\bullet(L_\lambda \boxtimes \mathrm{IC}_\lambda, L_\gamma \boxtimes \mathrm{IC}_\gamma) \quad (1.3)$$

inside A into the external tensor products of

$$\mathrm{hom}_{\mathbb{C}}(L_\lambda, L_\mu) \times \mathrm{hom}_{\mathbb{C}}(L_\mu, L_\gamma) \rightarrow \mathrm{hom}_{\mathbb{C}}(L_\lambda, L_\gamma), \text{ and} \quad (1.4)$$

$$\mathrm{Ext}_G^\bullet(\mathrm{IC}_\lambda, \mathrm{IC}_\mu) \times \mathrm{Ext}_G^\bullet(\mathrm{IC}_\mu, \mathrm{IC}_\gamma) \rightarrow \mathrm{Ext}_G^\bullet(\mathrm{IC}_\lambda, \mathrm{IC}_\gamma). \quad (1.5)$$

The associativity of (1.4) and (1.5) implies that of (1.3). By applying \mathbb{D} , we have identifications

$$\begin{aligned} \mathrm{hom}_{\mathbb{C}}(L_{\lambda}, L_{\mu}) &\cong \mathrm{hom}_{\mathbb{C}}(L_{\mu}^*, L_{\lambda}^*) \cong \mathrm{hom}_{\mathbb{C}}(L_{\mu^{\vee}}, L_{\lambda^{\vee}}), \text{ and} \\ \mathrm{Ext}_G^*(\mathrm{IC}_{\lambda}, \mathrm{IC}_{\mu}) &\cong \mathrm{Ext}_G^*(\mathbb{D}\mathrm{IC}_{\mu}, \mathbb{D}\mathrm{IC}_{\lambda}) \cong \mathrm{Ext}_G^*(\mathrm{IC}_{\mu^{\vee}}, \mathrm{IC}_{\lambda^{\vee}}) \end{aligned}$$

for each $\lambda, \mu \in \Lambda$. These identifications commute with the compositions (1.4) and (1.5). Taking direct sums, we find an algebra isomorphism $\psi : A \cong A^{op}$.

Since \mathbb{D} exchanges $L_{\lambda} \boxtimes \mathrm{IC}_{\lambda}$ with $L_{\lambda^{\vee}} \boxtimes \mathrm{IC}_{\lambda^{\vee}}$, we have

$$\begin{aligned} A \supset \mathrm{Ext}_G^{\bullet}(L_{\lambda} \boxtimes \mathrm{IC}_{\lambda}, L_{\lambda} \boxtimes \mathrm{IC}_{\lambda}) &\supset \mathrm{end}_{\mathbb{C}}(L_{\lambda}) \\ &\xrightarrow{\psi} \mathrm{end}_{\mathbb{C}}(L_{\lambda^{\vee}})^{op} \subset \mathrm{Ext}_G^{\bullet}(L_{\lambda^{\vee}} \boxtimes \mathrm{IC}_{\lambda^{\vee}}, L_{\lambda^{\vee}} \boxtimes \mathrm{IC}_{\lambda^{\vee}}) \subset A^{op}. \end{aligned}$$

Therefore, L_{λ}^* naturally carries the structure of a graded A -module $L_{\lambda^{\vee}}$ (via ψ) as required. \square

Lemma 1.3. *The algebra A is left and right Noetherian.*

Proof. Let C be the image of the natural map $H_G^{\bullet}(\mathrm{pt}) \rightarrow A$. Since $H_G^{\bullet}(\mathrm{pt})$ is a polynomial algebra, we deduce that C is a Noetherian algebra. By (1.2), A is a finitely generated module over C . Hence, every C -submodule of A is again finitely generated. This particularly applies to every left or right A -submodule of A (i.e. ideals of A), and hence the result. \square

For each $\lambda \in \Lambda$, we set

$$P_{\lambda} := \mathrm{Ext}_G^{\bullet}(\mathrm{IC}_{\lambda}, \mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}_G^i(\mathrm{IC}_{\lambda}, \mathcal{L}).$$

Each P_{λ} is a graded projective left A -module. By construction, we have

$$A \cong \bigoplus_{\lambda \in \Lambda} L_{\lambda}^* \boxtimes \mathrm{Ext}_G^{\bullet}(\mathrm{IC}_{\lambda}, \mathcal{L}) = \bigoplus_{\lambda \in \Lambda} P_{\lambda} \boxtimes L_{\lambda}^*$$

as left A -modules. It is standard that P_{λ} is an indecomposable A -module whose head is isomorphic to L_{λ} (cf. [CG97] §8.7). We have an idempotent $e_{\lambda} \in A$ so that $P_{\lambda} \cong Ae_{\lambda}$ as left graded A -modules (up to a grading shift).

For each $\lambda = (\underline{\lambda}, \xi) \in \Lambda$, we set

$$\tilde{K}_{\lambda} := \mathrm{Ext}_G^{\bullet}(\mathbb{C}_{\lambda}, \mathcal{L}) \text{ and } K_{\lambda} := \mathrm{Hom}_{C_{\lambda}}(\xi, H^{\bullet} i_{\lambda}^! \mathcal{L}[\dim \mathbb{O}_{\lambda}]).$$

We call K_{λ} a standard module, and \tilde{K}_{λ} a dual standard module of A . Here the Serre-type spectral sequence takes the form:

$$\begin{aligned} E_2 &\cong \mathrm{Ext}_G^{\bullet}(\xi[\dim \mathbb{O}_{\lambda}], j_{\lambda}^! \mathcal{L}) \\ &\cong \mathrm{Ext}_{G_{\lambda}}^{\bullet}(\xi[-\dim \mathbb{O}_{\lambda}], i_{\lambda}^! \mathcal{L}) \\ &\cong \mathrm{Ext}_{G_{\lambda}^{\circ}}^{\bullet + \dim \mathbb{O}_{\lambda}}(\xi, i_{\lambda}^! \mathcal{L})^{C_{\lambda}} \\ &\cong \bigoplus_{\mu=(\underline{\lambda}, \zeta) \in \Lambda} \mathrm{Hom}_{C_{\lambda}}(\xi, \zeta \otimes_{\mathbb{C}} H_{G_{\lambda}^{\circ}}^{\bullet}(\{x_{\lambda}\})) \boxtimes K_{\mu} \Rightarrow \tilde{K}_{\lambda}. \end{aligned} \quad (1.6)$$

Note that the first isomorphism is adjunction, the second is the induction equivalence (cf. [BL94] 2.6.2), the third is the Hochschild-Serre spectral sequence, and

the fourth is expanding $i_\lambda^! \mathcal{L}$ with the action of C_λ recorded and the usual Serre spectral sequence.

We now specialize to the case where our base field \mathbb{k} is the algebraic closure of a finite field \mathbb{F}_q of cardinality q . We regard each IC_λ as a simple mixed perverse sheaf (of some weight) in the category of mixed sheaves on \mathfrak{X} via [BBD82] §5.1.1, and each L_λ as a mixed (complex of) vector space of weight zero. In other words, we find a Frobenius endomorphism Fr of \mathfrak{X} and equip an isomorphism $\phi : \mathrm{Fr}^* \mathrm{IC}_\lambda \xrightarrow{\cong} \mathrm{IC}_\lambda$ that represents the \mathbb{F}_q -rational structure (see §2 for more detail). Moreover, the ϕ -action on L_λ^i is $q^{i/2} \mathrm{id}$ for each $i \in \mathbb{Z}$. It follows that the algebra A and its standard modules $\{K_\lambda\}_{\lambda \in \Lambda}$ acquire (mixed) weight structures (or the action of ϕ).

In addition, we also equip \mathbb{C}_λ (for $\lambda = (\lambda, \xi) \in \Lambda$) with a mixed weight structure. In particular, dual standard modules $\{\tilde{K}_\lambda\}_{\lambda \in \Lambda}$ also acquire some (mixed) weight structures.

Condition 1.4 (Condition (\clubsuit)). The condition (\clubsuit) consists of two subconditions:

- $(\clubsuit)_1$ The algebra A is pure of weight zero;
- $(\clubsuit)_2$ For each $\lambda \in \Lambda$, the sheaf IC_λ (whose weight is normalized to be zero) is pointwise pure of weight zero.

Remark 1.5. **1)** The conditions (\spadesuit) , and (\clubsuit) are closed under the restriction to a closed G -subvariety (cf. Lemma B.2); **2)** Thanks to Lusztig [Lu90a] and Varagnolo-Vasserot [VV99], the quiver Schur algebras of type A also satisfy the conditions (\spadesuit) and (\clubsuit) by taking \mathfrak{X} as the spaces of nilpotent representations of cyclic quivers.

Theorem 1.6. *We assume (\spadesuit) and $(\clubsuit)_1$, but not $(\clubsuit)_2$. We have $(\clubsuit)_2$ as a consequence of the following two conditions:*

- a) *The spectral sequence (1.6) is E_2 -degenerate for each $\lambda \in \Lambda$;*
- b) *For each $\lambda \in \Lambda$, we have a natural morphism $\psi_\lambda : H_G^\bullet(\mathrm{pt}) \rightarrow H_G^\bullet(\mathbb{O}_\lambda)$ of graded algebras. We have*

$$\ker \psi_\lambda \not\subset \ker \psi_\gamma \quad \text{for every} \quad \gamma \prec \lambda \in \Lambda;$$

Here the condition a) can be replaced by its variant:

- a)' *For each $\lambda, \mu \in \Lambda$, the stalk of IC_λ along \mathbb{O}_μ satisfies the parity vanishing.*

The proof of Theorem 1.6 is given in Appendix B.

Remark 1.7. **1)** The condition $(\clubsuit)_2$ implies the condition a) of Theorem 1.6 since $H_M^\bullet(\mathrm{pt})$ is pure for an affine algebraic group M ; **2)** Theorem 1.6 presents new proofs of pointwise purity of (equivariant) intersection cohomology complexes on some varieties from the structure of the affine Hecke algebras of type BC with 2-parameters [K09], and the quiver Schur algebras [Lu90a, VV99].

For $M \in A\text{-gmod}$ and $i \in \mathbb{Z}$, we define

$$\begin{aligned} [M : L_\lambda \langle i \rangle]_0 &:= \dim \mathrm{Hom}_{A\text{-gmod}}(P_\lambda \langle i \rangle, M) \in \mathbb{Z} \quad \text{and} \\ [M : L_\lambda] &:= \mathrm{gdim} \mathrm{hom}_A(P_\lambda, M) \in \mathbb{Z}((t)). \end{aligned}$$

We have $[M : L_\lambda] = \sum_{i \in \mathbb{Z}} [M : L_\lambda \langle i \rangle]_0 t^i \in \mathbb{Z}((t))$. This is a graded version of the composition multiplicity of M , that is existent as A is finite-dimensional modulo its graded Jacobson radical (cf. Lemma 1.12 in the below).

We define $A\text{-gmod}^{pf}$ to be the full subcategory of $A\text{-gmod}$ consisting of objects which admit finite resolutions by finitely generated graded projective A -modules (this is an additive category). For $M \in A\text{-gmod}^{pf}$ and $N \in A\text{-gmod}$, we define its graded Euler-Poincaré characteristic as:

$$\langle M, N \rangle_{\text{gEP}} := \sum_{i \geq 0} (-1)^i \text{gdim} \text{ext}_A^i(M, N) \in \mathbb{Z}((t)). \quad (1.7)$$

Theorem 1.8. *Assume the conditions (\spadesuit) and (\clubsuit):*

1. *We have*

$$[\tilde{K}_\lambda : L_\mu] = 0 = [K_\lambda : L_\mu] \quad \text{for } \lambda \not\sim \mu \quad \text{and} \quad [K_\lambda : L_\mu] = \delta_{\lambda, \mu} \quad \text{for } \lambda \sim \mu;$$

2. *We have*

$$\text{ext}_A^\bullet(\tilde{K}_\lambda, \tilde{K}_\mu) = \{0\} \quad \text{for each } \mu \not\sim \lambda \quad \text{and} \quad \text{ext}_A^\bullet(\tilde{K}_\lambda, K_\mu) = \{0\} \quad \text{for each } \mu \not\sim \lambda;$$

3. *For each $\lambda \in \Lambda$, we have*

$$\tilde{K}_\lambda \cong P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right);$$

4. *Each \tilde{K}_λ admits a separable decreasing filtration whose associated graded is a direct sum of grading shifts of K_μ with $\mu \sim \lambda$. In addition, we have*

$$[\tilde{K}_\lambda : L_\mu] = \text{gdim} \text{hom}_{C_\lambda}(\xi, \zeta \otimes_{\mathbb{C}} H_{G_\lambda}^\bullet(\text{pt}))$$

$$\text{for every } \lambda = (\underline{\lambda}, \xi) \sim (\underline{\mu}, \zeta) = \mu.$$

Remark 1.9. Theorem 1.8 still hold if some of L_λ is zero, but we need to impose the following condition:

- For each $\lambda \in \Lambda$ with $L_\lambda \neq \{0\}$, every restriction of IC_λ to a G -orbit is a direct sum of local systems corresponding to $\mu \in \Lambda$ with $L_\mu \neq \{0\}$.

This is the situation we encounter in [K15] (cf. [Ach13]), and all the results in sections three and four still hold with straight-forward modifications.

The proof of Theorem 1.8 is given at the end of section two.

Corollary 1.10 (of Theorem 1.8). *The matrices $K := ([K_\lambda : L_\mu])_{\lambda, \mu \in \Lambda}$ and $\tilde{K} := ([\tilde{K}_\lambda : L_\mu])_{\lambda, \mu \in \Lambda}$ are blockwise upper-triangular and invertible in $\mathbb{Q}((t))$.*

Proof. Thanks to Theorem 1.8 1), K satisfies the desired property. Here we also deduce that \tilde{K} is blockwise upper-triangular. The block-diagonal entries of \tilde{K} are invertible since they are identity modulo $t\mathbb{Z}[[t]]$ by Theorem 1.8 4). Therefore, we can invert the matrix \tilde{K} as desired. \square

Lemma 1.11. *The graded algebra $A_{(G, \mathfrak{x})}$ is graded Morita equivalent to $B_{(G, \mathfrak{x})}$.*

Proof. By construction, we see that

$$[P_\lambda : L_\mu] = \text{gdim Ext}_G^\bullet(\text{IC}_\lambda, \text{IC}_\mu) \quad \text{for each } \lambda, \mu \in \Lambda.$$

In particular, the graded vector space $\text{hom}_A(P_\mu, P_\lambda)$ does not depend on the choice of \mathcal{L} . Since the (graded) algebra structure of A arises from the Yoneda compositions, the composition map

$$\text{hom}_A(P_\gamma, P_\mu) \times \text{hom}_A(P_\mu, P_\lambda) \longrightarrow \text{hom}_A(P_\gamma, P_\lambda) \quad \text{for each } \lambda, \mu, \gamma \in \Lambda$$

is identified with the composition map

$$\text{Ext}_G^\bullet(\text{IC}_\lambda, \text{IC}_\mu) \times \text{Ext}_G^\bullet(\text{IC}_\mu, \text{IC}_\gamma) \longrightarrow \text{Ext}_G^\bullet(\text{IC}_\lambda, \text{IC}_\gamma).$$

Therefore, we deduce $\text{end}_A(\bigoplus_{\lambda \in \Lambda} P_\lambda)^{op} \cong B_{(G, \mathfrak{x})}$. Hence, $\text{hom}_A(\bigoplus_{\lambda \in \Lambda} P_\lambda, \bullet)$ yields the desired graded Morita equivalence through Lemma 1.2. \square

Lemma 1.12. *A graded A -module M with finite dimensional graded pieces is generated by its head if the grading of M is bounded from the below.*

Proof. Let P be the projective cover of $\text{hd } M$, that is the maximal graded semisimple quotient of M . Let us denote the lifting map $P \rightarrow M$ by ϕ . We set $N := \text{coker } \phi$. Since a simple quotient of N is a simple quotient of M , we conclude that $\text{hd } N = \{0\}$. As a quotient of M , the grading of N is bounded from the below. Since $\text{Im } \phi$ is precisely the submodule of M spanned by $\text{hd } M$, we assume that $N \neq \{0\}$ to deduce contradiction.

Let $c := \min\{k \in \mathbb{Z} \mid A^k \neq \{0\}\}$ and let $b := \min\{k \in \mathbb{Z} \mid N^k \neq \{0\}\}$. Consider the graded A -submodule N_c of N generated by $\{N^k\}_{k > b-c}$. Then, we have $N_c^b = \{0\}$ by a degree counting. In particular, we have $N/N_c \neq \{0\}$. Since $N^k = N_c^k$ for $k > b - c$ and each graded piece of M is finite-dimensional, we deduce that N/N_c is finite-dimensional.

A finite-dimensional graded A -module admits a simple graded quotient since the action of A factors through a finite-dimensional algebra, which is Artin. As a simple quotient of N/N_c is a simple quotient of N , we have a contradiction and hence $\text{Im } \phi = M$ as required. \square

Corollary 1.13. *Suppose that $A^i = \{0\}$ for $i < c$. Let M be a graded A -module with finite dimensional graded pieces so that $M^i = \{0\}$ for $i < b$. Then, M admits a minimal graded projective A -resolution (that are not necessarily of finite length) so that its k -th term P_k satisfies $P_k^i = \{0\}$ for $i < b + (k+1)c$ and has finite-dimensional graded pieces.*

Proof. We have the projective cover P_0 of M by lifting $\text{hd } M$, whose grading ranges the sum of that of A and M . Hence, we have $P_0^i = \{0\}$ for $i < b + c$. In addition, we have

$$\dim P_0^i \leq \sum_{j \in \mathbb{Z}} \dim M^j \cdot \dim A^{i-j} < \infty \quad \text{for each } i \in \mathbb{Z}.$$

Assume that we have a minimal projective resolution of M with the desired property up to the k -th term:

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} P_{k-2} \cdots P_1 \xrightarrow{d_1} P_0 \rightarrow M.$$

We set P_{k+1} to be the projective cover of $\mathrm{hd} \ker d_k$ and d_{k+1} to be its lift to P_k to construct a projective resolution up to the $(k+1)$ -th term. Since $P_k^i = \{0\}$ for $i < b + (k+1)c$, we deduce $(\ker d_k)^i = \{0\}$ for $i < b + (k+1)c$ and $P_{k+1}^i = \{0\}$ for $i < b + (k+2)c$. In addition, Lemma 1.12 asserts that P_{k+1} surjects onto $\ker d_k$, and hence P_{k+1} gives the $(k+1)$ -th term of a projective resolution of M . Since $\ker d_k$ and A have finite-dimensional graded pieces and their gradings are bounded from the below, we deduce that each graded piece of P_{k+1} is also finite-dimensional. As the minimality assumption yields that the induced map $\mathrm{hd} \ker d_k \rightarrow P_k \rightarrow \mathrm{hd} P_k$ is zero, our complex must be minimal. This proceeds the induction (which might not terminate), and hence the result. \square

2 Standard complexes and a proof of Theorem 1.8

In this section, we employ the following settings:

Let \mathbb{k} be an algebraic closure of a finite field. Let \mathfrak{X} be a variety over \mathbb{k} equipped with an action of an affine algebraic group G (over \mathbb{k}). We fix some sufficiently large prime p and its power q so that there exists a variety $\mathfrak{X}_{\mathbb{F}_q}$ and an affine algebraic group $G_{\mathbb{F}_q}$ over \mathbb{F}_q such that (G, \mathfrak{X}) is the scalar extension of $(G_{\mathbb{F}_q}, \mathfrak{X}_{\mathbb{F}_q})$ from \mathbb{F}_q to \mathbb{k} (through $\mathbb{Z}_S \rightsquigarrow \mathbb{F}_q \rightsquigarrow \mathbb{k}$). Let Fr be the Frobenius morphism defined over \mathbb{F}_q (on \mathfrak{X} , some model of EG , or $EG \times^G \mathfrak{X}$).

We say that an object $\mathcal{E} \in D_G^b(\mathfrak{X})$ is equipped with a mixed structure if we specify an isomorphism $\phi : \mathrm{Fr}^* \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ that descends \mathcal{E} to an object $\mathcal{E}_{\mathbb{F}_q} \in D_{G_{\mathbb{F}_q}}^b(\mathfrak{X}_{\mathbb{F}_q})$ that defines a mixed complex on $\mathfrak{X}_{\mathbb{F}_q}$ (cf. [BBD82] §5.1.1 and §5.1.5). We sometimes omit ϕ if the meaning is clear from the context.

For two objects $\mathcal{E}, \mathcal{F} \in D_G^b(\mathfrak{X})$ with their mixed structures $\phi_{\mathcal{E}}, \phi_{\mathcal{F}}$ and $n \in \mathbb{Z}$, we have an induced action of the Frobenius morphism on $\mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F})$ defined as

$$\mathrm{F} : \mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F}) \ni f \mapsto (\phi_{\mathcal{F}} \otimes^! \mathbb{D}(\phi_{\mathcal{E}})) \circ \mathrm{Fr}^*(f) \in \mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F}).$$

This defines the pure-part (of weight zero) of $\mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F})$ as

$$\mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F})_{\mathrm{pure}} := \{\xi \in \mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F}) \mid \exists \alpha \in \overline{\mathbb{Q}}, \mathrm{F}(\xi) = \alpha \xi, |\alpha| = q^{n/2}\} \subset \mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F}).$$

By setting $\mathcal{E} = \mathbb{C}$, we also have the F -action on the cohomologies.

Remark 2.1. 1) The i -th cohomology $H^i(\mathcal{S})$ of a constructible complex of $\overline{\mathbb{Q}}_\ell$ -sheaves \mathcal{S} with a mixed structure (over $\mathrm{Spec} \mathbb{k}$) has the absolute value of its F -eigenvalue $\leq q^{(a+i)/2}$ (resp. $\geq q^{(a+i)/2}$) for each $i \in \mathbb{Z}$ if and only if it has weight $\leq a$ (resp. $\geq a$) as a constructible complex of $\overline{\mathbb{Q}}_\ell$ -sheaves over $\mathrm{Spec} \mathbb{F}_q$ in the sense of [BBD82] 5.1.5 and 5.1.8. By [BBD82] 5.1.14, our cohomologies and Hom-spaces (over $\mathfrak{X}_{\mathbb{k}}$) admits the F -eigenvalue estimate as in [BBD82] §5.1.13, that we sometimes refer as a weight estimate (cf. Deligne [Del81]).

2) We have a natural forgetful functor

$$D_{G_{\mathbb{F}_q}}^b(\mathfrak{X}_{\mathbb{F}_q}) \longrightarrow D_G^b(\mathfrak{X}).$$

For each $\mathcal{E}, \mathcal{F} \in D_{G_{\mathbb{F}_q}}^b(\mathfrak{X}_{\mathbb{F}_q})$ and $n \in \mathbb{Z}$, we have a map of \mathbb{C} -vector spaces:

$$\varkappa : \mathrm{Ext}_{D_{G_{\mathbb{F}_q}}^b(\mathfrak{X}_{\mathbb{F}_q})}^n(\mathcal{E}, \mathcal{F}) \rightarrow \mathrm{Ext}_G^n(\mathcal{E}, \mathcal{F}). \quad (2.1)$$

The map \varkappa is neither injection nor surjection in general since the LHS respects the information on the Frobenius endomorphism imposed on objects (module structures/extensions) while the RHS carries a $\mathbb{C}[\mathbb{F}]$ -module structure inherited from the objects. Taking the Galois cohomology on the RHS yields a spectral sequence converging to the LHS (see [BBD82] (5.1.2.3)). Note that the image of \varkappa is contained in $\mathrm{Hom}_G(\mathcal{E}, \mathcal{F})_{\mathrm{pure}}$ when $n = 0$ by [BBD82] (5.1.2.5). In particular, our construction in the below basically concerns with the image of \varkappa , that involves some sort of “semi-simplification operation” of the Frobenius action at each stage viewed from a treatment using a mixed version of the equivariant derived category.

We begin by general results on weight filtration, for which some part seems to follow from [Bon12, Pau08, Sch11b].

Definition 2.2 (Perverse class). For $\mathcal{E} \in D_G^b(\mathfrak{X})$, we define

$$[\mathcal{E}]_t := \sum_{m \in \mathbb{Z}} [{}^p H^m(\mathcal{E})] \otimes t^m \in K(\mathrm{Perv}_G \mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}],$$

where $\mathrm{Perv}_G \mathfrak{X}$ is the category of G -equivariant perverse sheaves on \mathfrak{X} (cf. [BL94] 5.1), $K(\mathrm{Perv}_G \mathfrak{X})$ is its Grothendieck group, and $[\mathcal{F}]$ in the RHS denote the class of $\mathcal{F} \in \mathrm{Perv}_G \mathfrak{X}$ in $K(\mathrm{Perv}_G \mathfrak{X})$.

It is straight-forward to see $[\mathcal{E}]_t = [\tau_{<m} \mathcal{E}]_t + [\tau_{\geq m} \mathcal{E}]_t$ for each $m \in \mathbb{Z}$.

Lemma 2.3. *For each $\mathcal{E} \in D_G^b(\mathfrak{X})$, we have $\#\{m \in \mathbb{Z} \mid {}^p H^m(\mathcal{E}) \neq \{0\}\} < \infty$. In particular, the perverse classes are well-defined elements of $K(\mathrm{Perv}_G \mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$.*

Proof. Since \mathcal{E} is constructible, its stalk and costalk are bounded complexes at each point of \mathcal{X} . Thus, the characterization of objects of $\tau_{\geq 0} D^b(\mathfrak{X})$ and $\tau_{\leq 0} D^b(\mathfrak{X})$ in terms of stalks and costalks [BBD82] 2.2.2 implies the result (as there are finitely many stratum where the stalks/costalks of \mathcal{E} are constant). \square

Lemma 2.4. *Let $\mathcal{E} \in D_G^b(\mathfrak{X})$ be a complex equipped with a mixed structure. For each $a \in \mathbb{Z}$, there exist complexes $F_{<a} \mathcal{E}, F_{\geq a} \mathcal{E} \in D_G^b(\mathfrak{X})$ equipped with mixed structures so that the following properties hold:*

1. $F_{<a} \mathcal{E}$ has weight $< a$, and $F_{\geq a} \mathcal{E}$ has weight $\geq a$;
2. We have a distinguished triangle

$$F_{<a} \mathcal{E} \rightarrow \mathcal{E} \rightarrow F_{\geq a} \mathcal{E} \xrightarrow{+1} \quad \text{in } D_G^b(\mathfrak{X}), \quad (2.2)$$

where the morphisms respect the mixed structures;

3. We have $[\mathcal{E}]_t = [F_{<a} \mathcal{E}]_t + [F_{\geq a} \mathcal{E}]_t \in K(\mathrm{Perv}_G \mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$;
4. Assume that $\mathcal{G}, \mathcal{H} \in D_G^b(\mathfrak{X})$ are complexes equipped with mixed structures so that the pair $(\mathcal{G}, \mathcal{H})$ satisfies the same properties as $(F_{<a} \mathcal{E}, F_{\geq a} \mathcal{E})$ described in the above.
 - We have isomorphisms $\mathcal{G} \cong F_{<a} \mathcal{E}$ and $\mathcal{H} \cong F_{\geq a} \mathcal{E}$ in $D_G^b(\mathfrak{X})$ (without mixed structures);

- We have ${}^pH^i(\mathcal{G}) \cong {}^pH^i(F_{< a}\mathcal{E})$ and ${}^pH^i(\mathcal{H}) \cong {}^pH^i(F_{\geq a}\mathcal{E})$ as G -equivariant mixed perverse sheaves for each $i \in \mathbb{Z}$.

Proof. We first construct complexes $F_{< a}\mathcal{E}$ and $F_{\geq a}\mathcal{E}$ which satisfies the the first three properties.

For each $m \in \mathbb{Z}$, the weight $< (a + m)$ -part ${}^pH^m(\mathcal{E})_{< a}$ of ${}^pH^m(\mathcal{E})$ is a subobject by [BBD82] 5.3.5. By induction on m , we construct distinguished triangles

$$\mathcal{E}_m^{< a} \rightarrow \tau_{> m}\mathcal{E} \rightarrow \mathcal{E}_m^{\geq a} \xrightarrow{+1} \quad \text{in } D_G^b(\mathfrak{X}), \quad (2.3)$$

with the following properties:

(2.3)₀ Each morphism of (2.3) intertwines the mixed structures;

(2.3)₁ $\mathcal{E}_m^{< a}$ has weight $< a$, and $\mathcal{E}_m^{\geq a}$ has weight $\geq a$;

(2.3)₂ We have $[\tau_{> m}\mathcal{E}]_t = [\mathcal{E}_m^{< a}]_t + [\mathcal{E}_m^{\geq a}]_t$;

(2.3)₃ We have $\tau_{\leq m}\mathcal{E}_m^{< a} \cong \{0\} \cong \tau_{\leq m}\mathcal{E}_m^{\geq a}$.

We assume that a distinguished triangle of the form (2.3) for m exists and construct a distinguished triangle of the form (2.3) for $m - 1$. We have $\tau_{> m}\mathcal{E} = \{0\}$ for $m \gg 0$, and hence the case $m \gg 0$ is verified by setting $\mathcal{E}_m^{< a} = \{0\} = \mathcal{E}_m^{\geq a}$. By the existence of the weight filtration of mixed perverse sheaves (cf. [BBD82] 5.3.5), we deduce that the (shifted) complex $H_{\geq a}^m := ({}^pH^m(\mathcal{E})/{}^pH^m(\mathcal{E})_{< a})[-m]$ has weight $\geq a$. Here, $\tau_{\geq m}\mathcal{E}$ is determined by the map $\tau_{> m}\mathcal{E} \rightarrow {}^pH^m(\mathcal{E})[1 - m]$. Its pullback to $\mathcal{E}_m^{< a}$ factors through ${}^pH^m(\mathcal{E})_{< a}[1 - m]$ since

$$\text{Hom}_G(\mathcal{E}_m^{< a}, H_{\geq a}^m[1])_{\text{pure}} = \{0\}$$

by [BBD82] 5.1.15. In addition, [BBD82] 5.1.15 ii) claims that the corresponding Hom-space vanishes over \mathbb{F}_q . Hence, [BBD82] 1.1.11 yields the following commutative diagram of six distinguished triangles (that is compatible with mixed structures):

$$\begin{array}{ccccc} \mathcal{E}_m^{< a}[-1] & \longrightarrow & (\tau_{> m}\mathcal{E})[-1] & \longrightarrow & \mathcal{E}_m^{\geq a}[-1] \xrightarrow{+1} \\ \downarrow +1 & & \downarrow +1 & & \downarrow +1 \\ {}^pH^m(\mathcal{E})_{< a}[-m] & \longrightarrow & {}^pH^m(\mathcal{E})[-m] & \longrightarrow & H_{\geq a}^m \xrightarrow{+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_{m-1}^{< a} & \longrightarrow & \tau_{\geq m}\mathcal{E} & \longrightarrow & \mathcal{E}_{m-1}^{\geq a} \xrightarrow{+1} \\ \downarrow & & \downarrow & & \downarrow \end{array} \quad (2.4)$$

Here $\mathcal{E}_{m-1}^{< a}$ and $\mathcal{E}_{m-1}^{\geq a}$ are determined by the column maps. By construction, $\mathcal{E}_{m-1}^{< a}$ has weight $< a$ and $\mathcal{E}_{m-1}^{\geq a}$ has weight $\geq a$. Therefore, we obtain a distinguished triangle satisfying (2.3)₀ and (2.3)₁ for $(m - 1)$.

Claim A. We have the following identities:

$$\begin{aligned} [\mathcal{E}_{m-1}^{< a}]_t &= [\mathcal{E}_m^{< a}]_t + [{}^pH^m(\mathcal{E})_{< a}] \otimes t^m, & \tau_{\leq m-1}\mathcal{E}_{m-1}^{< a} &= \{0\} \\ [\mathcal{E}_{m-1}^{\geq a}]_t &= [\mathcal{E}_m^{\geq a}]_t + [H_{\geq a}^m]_t, & \tau_{\leq m-1}\mathcal{E}_{m-1}^{\geq a} &= \{0\} \\ [\tau_{\geq m}\mathcal{E}]_t &= [\tau_{> m}\mathcal{E}]_t + [{}^pH^m(\mathcal{E})] \otimes t^m, & [{}^pH^m(\mathcal{E})] \otimes t^m &= [{}^pH^m(\mathcal{E})_{< a}] \otimes t^m + [H_{\geq a}^m]_t. \end{aligned}$$

Proof. We have ${}^pH^i({}^pH^m(\mathcal{E})_{<a}[-m]) = {}^pH^m(\mathcal{E})_{<a}$ ($i = m$) or $\{0\}$ ($i \neq m$) by construction. Thus, the long exact sequence of perverse cohomologies of the first column of (2.4) reads as:

$$\begin{aligned} 0 \rightarrow {}^pH^m(\mathcal{E})_{<a} &\xrightarrow{\cong} {}^pH^m(\mathcal{E}_{m-1}^{<a}) \rightarrow 0 \\ &\rightarrow 0 \rightarrow {}^pH^{m+1}(\mathcal{E}_{m-1}^{<a}) \rightarrow {}^pH^{m+1}(\mathcal{E}_m^{<a}) \rightarrow 0 \rightarrow \dots \end{aligned}$$

This implies the first two identities. By a similar analysis applied to the third column of (2.4), we conclude the third and fourth identities. The fifth identity follows from the definition of the perverse class. The sixth identity follows from the definition of $H_{\geq a}^m$. \square

We return to the proof of Lemma 2.4. By Claim A and (2.3)₂ for m , we have

$$\begin{aligned} [\mathcal{E}_{m-1}^{<a}]_t + [\mathcal{E}_{m-1}^{\geq a}]_t &= [\mathcal{E}_m^{<a}]_t + [\mathcal{E}_m^{\geq a}]_t + [{}^pH^m(\mathcal{E})_{<a}] \otimes t^m + [H_{\geq a}^m]_t \\ &= [\mathcal{E}_m^{<a}]_t + [\mathcal{E}_m^{\geq a}]_t + [{}^pH^m(\mathcal{E})] \otimes t^m \\ &= [\tau_{>m}\mathcal{E}]_t + [{}^pH^m(\mathcal{E})] \otimes t^m = [\tau_{\geq m}\mathcal{E}]_t. \end{aligned}$$

This verifies that the bottom distinguished triangle of (2.4) satisfies (2.3)₂ for $m-1$. In addition, it also satisfies (2.3)₃ for $m-1$ by Claim A. Therefore, we obtain a distinguished triangle of type (2.3) for $m-1$. By Lemma 2.3, we know ${}^pH^m(\mathcal{E}) = \{0\}$ for $m \ll 0$. Therefore, we take $m \ll 0$ so that $\mathcal{E} \cong \tau_{>m}\mathcal{E}$ and set $F_{<a}\mathcal{E} := \mathcal{E}_m^{<a}$ and $F_{\geq a}\mathcal{E} := \mathcal{E}_m^{\geq a}$ to obtain complexes with the first three properties.

Now we prove the fourth property. We have morphisms $\iota : F_{<a}\mathcal{E} \rightarrow \mathcal{E}, j : \mathcal{E} \rightarrow F_{\geq a}\mathcal{E}$ and morphisms $\iota' : \mathcal{G} \rightarrow \mathcal{E}, j' : \mathcal{E} \rightarrow \mathcal{H}$ in the distinguished triangle (2.2) and its analogue for $(\mathcal{G}, \mathcal{H})$. These morphisms intertwine mixed structures by assumption.

The composition morphism $j \circ \iota'$ belongs to $\text{Hom}_G(\mathcal{G}, F_{\geq a}\mathcal{E})_{\text{pure}}$ as it respects the mixed structures. By the weight comparison (cf. [BBD82] 5.1.14), the vector space $\text{Hom}_G(\mathcal{G}, F_{\geq a}\mathcal{E})$ has weight > 0 . In particular, we have

$$\{0\} = \text{Hom}_G(\mathcal{G}, F_{\geq a}\mathcal{E})_{\text{pure}} \subset \text{Hom}_G(\mathcal{G}, F_{\geq a}\mathcal{E}).$$

Therefore, the morphism ι' factors through $F_{<a}\mathcal{E}$. This defines a morphism $\theta : \mathcal{G} \rightarrow F_{<a}\mathcal{E}$.

Hence, we have a morphism ϑ (coming from the triangulated structure of $D_G^b(\mathfrak{X})$, and is not necessarily respect the mixed structures) which fits into:

$$\begin{array}{ccccccc} \mathcal{G} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{H} & \xrightarrow{+1} & \longrightarrow \\ \downarrow \theta & & \downarrow = & & \downarrow \vartheta & & \\ F_{<a}\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & F_{\geq a}\mathcal{E} & \xrightarrow{+1} & \longrightarrow \end{array}$$

By taking their long exact sequences, we find a commutative diagram in the (abelian) category of perverse sheaves on \mathfrak{X}

$$\begin{array}{ccccccc} \longrightarrow & {}^pH^k(\mathcal{G}) & \longrightarrow & {}^pH^k(\mathcal{E}) & \longrightarrow & {}^pH^k(\mathcal{H}) & \longrightarrow {}^pH^{k+1}(\mathcal{G}) \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \longrightarrow & {}^pH^k(F_{<a}\mathcal{E}) & \longrightarrow & {}^pH^k(\mathcal{E}) & \longrightarrow & {}^pH^k(\mathcal{H}) & \longrightarrow {}^pH^k(F_{\geq a}\mathcal{E}) \longrightarrow \dots \end{array}$$

By assumption, we have equalities

$$[{}^p H^k(F_{<a}\mathcal{E})] + [{}^p H^k(F_{\geq a}\mathcal{E})] = [{}^p H^k(\mathcal{E})] = [{}^p H^k(\mathcal{G})] + [{}^p H^k(\mathcal{H})] \quad \text{for each } k \in \mathbb{Z}.$$

These equalities, together with Lemma 2.3, forces the above commutative diagram to split into commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}^p H^k(\mathcal{G}) & \longrightarrow & {}^p H^k(\mathcal{E}) & \longrightarrow & {}^p H^k(\mathcal{H}) \longrightarrow 0 \\ & & \downarrow {}^p H^k(\theta) & & \downarrow = & & \downarrow {}^p H^k(\vartheta) \\ 0 & \longrightarrow & {}^p H^k(F_{<a}\mathcal{E}) & \longrightarrow & {}^p H^k(\mathcal{E}) & \longrightarrow & {}^p H^k(F_{\geq a}\mathcal{E}) \longrightarrow 0 \end{array}$$

for (all) $k \in \mathbb{Z}$.

In addition, ${}^p H^k(F_{<a}\mathcal{E})$ and ${}^p H^k(\mathcal{G})$ have weight $< a$, and ${}^p H^k(F_{\geq a}\mathcal{E})$ and ${}^p H^k(\mathcal{H})$ have weight $\geq a$. It follows that the image of the both of ${}^p H^k(F_{<a}\mathcal{E})$ and ${}^p H^k(\mathcal{G})$ in ${}^p H^k(\mathcal{E})$ is the weight $< (a+k)$ -part of ${}^p H^k(\mathcal{E})$ (as their composition multiplicities are equal in the category of perverse sheaves).

Therefore, ${}^p H^k(\theta)$ is an isomorphism for each $k \in \mathbb{Z}$. By the five lemma, ${}^p H^k(\vartheta)$ is also an isomorphism for each $k \in \mathbb{Z}$. In particular, the mapping cone of each of θ and ϑ yields a complex with vanishing perverse cohomologies. It follows that θ and ϑ are isomorphisms that intertwines ι, ι' and j, j' , respectively.

In addition, the construction shows that ${}^p H^k(\mathcal{G}) = {}^p H^k(F_{<a}\mathcal{E})$ as a (mixed) subsheaf of ${}^p H^k(\mathcal{E})$, and ${}^p H^k(\mathcal{H}) \cong {}^p H^k(F_{\geq a}\mathcal{E})$ as (mixed) quotient sheaves of ${}^p H^k(\mathcal{E})$ for each $k \in \mathbb{Z}$. This finishes the proof the fourth property, and hence the entire proof. \square

Corollary 2.5. *Keep the setting of Lemma 2.4. For each $a \leq b$, we have*

$$F_{<a}(F_{<b}\mathcal{E}) \cong F_{<a}\mathcal{E}, \text{ and } F_{\geq b}(F_{\geq a}\mathcal{E}) \cong F_{\geq b}\mathcal{E}.$$

Proof. We use the setting of the proof of Lemma 2.4. For each $m \in \mathbb{Z}$, we have natural inclusions

$$F_{<(a+m)}H^m(\mathcal{E}) = F_{<(a+m)}F_{<(b+m)}H^m(\mathcal{E}) \subset F_{<(b+m)}H^m(\mathcal{E}) \subset H^m(\mathcal{E})$$

in the category of (G -equivariant mixed) perverse sheaves. Then, the pullback of $\tau_{>m}\mathcal{E} \rightarrow {}^p H^m(\mathcal{E})[1-m]$ to $\mathcal{E}_m^{<a}$ also factors through $(F_{<(b+m)}H^m(\mathcal{E}))[1-m]$, which induces a map $\mathcal{E}_{m-1}^{<a} \rightarrow \mathcal{E}_{m-1}^{<b}$ in case the natural map $\mathcal{E}_m^{<a} \rightarrow \tau_{>m}\mathcal{E}$ factors through $\mathcal{E}_m^{<b}$. It further induces an isomorphism $\mathcal{E}_{m-1}^{<a} \cong F_{<a}\mathcal{E}_{m-1}^{<b}$ provided if $\mathcal{E}_m^{<a} \cong F_{<a}\mathcal{E}_m^{<b}$. Hence, we deduce $\mathcal{E}_m^{<a} \cong F_{<a}\mathcal{E}_m^{<b}$ by a downward induction on m (since $\tau_{>m}\mathcal{E} = \{0\}$ for $m \gg 0$). This implies $F_{<a}(F_{<b}\mathcal{E}) \cong F_{<a}\mathcal{E}$. The assertion $F_{\geq b}(F_{\geq a}\mathcal{E}) \cong F_{\geq b}\mathcal{E}$ follows by applying the construction of $F_{<b}$ to the distinguished triangle $F_{<a}\mathcal{E} \rightarrow \mathcal{E} \rightarrow F_{\geq a}\mathcal{E} \xrightarrow{+1}$. \square

Lemma 2.6. *Let $\mathcal{E} \in D_G^b(\mathfrak{X})$ be a complex with a mixed structure. Then, there exists a collection of objects $\{F_{\geq a}\mathcal{E}\}_{a \in \mathbb{Z}}$ of $D_G^b(\mathfrak{X})$ with the following properties:*

1. Each $F_{\geq a}\mathcal{E}$ is of weight $\geq a$;
2. $F_{\geq a}\mathcal{E} = \{0\}$ for $a \gg 0$, and $F_{\geq a}\mathcal{E} \cong \mathcal{E}$ for $a \ll 0$;

3. For each $a \in \mathbb{Z}$, we have a distinguished triangle

$$\longrightarrow \mathrm{gr}_a \mathcal{E} \rightarrow F_{\geq a} \mathcal{E} \rightarrow F_{\geq (a+1)} \mathcal{E} \xrightarrow{+1} \quad \text{in } D_G^b(\mathfrak{X}), \quad (2.5)$$

where $\mathrm{gr}_a \mathcal{E}$ is a pure complex of weight a ;

4. We have $[F_{\geq a} \mathcal{E}]_t = \sum_{b \geq a} [\mathrm{gr}_b \mathcal{E}]_t$ and $[\mathcal{E}]_t = \sum_a [\mathrm{gr}_a \mathcal{E}]_t$;

5. Each $\mathrm{gr}_a \mathcal{E}$ is a direct sum of shifted simple G -equivariant perverse sheaves (if we discard its mixed structure).

Proof. For $a \ll 0$, we have $F_{\geq a} \mathcal{E} \cong \mathcal{E}$. For each $a \in \mathbb{Z}$,

$$F_{<(a+1)}(F_{\geq a} \mathcal{E}) \rightarrow F_{\geq a} \mathcal{E} \rightarrow F_{\geq (a+1)} \mathcal{E} \xrightarrow{+1}$$

is a distinguished triangle by Corollary 2.5. By construction, each $F_{<(a+1)}(F_{\geq a} \mathcal{E})$ must be pure of weight a and is non-zero only for a finitely many $a \in \mathbb{Z}$. By setting $\mathrm{gr}_a \mathcal{E} := F_{<(a+1)}(F_{\geq a} \mathcal{E})$, we deduce the first four assertions.

The fifth assertion is automatic for a(n arbitrary) pure complex in $D_G^b(\mathfrak{X})$ by [BBD82] 5.3.9 i) and 5.4.5. \square

Corollary 2.7. *In addition to our setting, we also employ the notation and setting of section one. In particular, the G -action on \mathfrak{X} has finitely many orbits. For each $\lambda = (\underline{\lambda}, \xi) \in \Lambda$, we arrange IC_λ and $\underline{\xi}[\dim \mathbb{O}_\lambda]$ to be pure of weight zero (cf. (1.1)). There exists a collection of objects $\{F_{\geq a} \mathbb{C}_\lambda\}_{a \in \mathbb{Z}}$ of $D_G^b(\mathfrak{X})$ with the following properties:*

1. We have $F_{\geq 0} \mathbb{C}_\lambda = \mathrm{IC}_\lambda$ and $F_{\geq a} \mathbb{C}_\lambda = \mathbb{C}_\lambda$ for $a \ll 0$;
2. For each $a < 0$, we have a distinguished triangle

$$\longrightarrow \mathrm{gr}_a \mathbb{C}_\lambda \rightarrow F_{\geq a} \mathbb{C}_\lambda \rightarrow F_{\geq (a+1)} \mathbb{C}_\lambda \xrightarrow{+1} \quad \text{in } D_G^b(\mathfrak{X}),$$

where $\mathrm{gr}_a \mathbb{C}_\lambda$ is a direct sum of $\{\mathrm{IC}_\mu[i]\}_{\mu \prec \lambda, i \in \mathbb{Z}}$;

3. For each $a \in \mathbb{Z}$, $F_{\geq a} \mathbb{C}_\lambda$ has weight $\geq a$ and $\mathrm{gr}_a \mathbb{C}_\lambda$ is pure of weight a .

Proof. We apply the construction of Lemma 2.6 for $\mathcal{E} := \mathbb{C}_\lambda$. The latter part of the first assertion is automatic and the third assertion is Lemma 2.6 5).

The operation $(j_\lambda)_!$ does not increase weights (cf. [BBD82] 5.1.14). In particular, we have $F_{>0} \mathbb{C}_\lambda = \{0\}$. Hence, $F_{\geq 0} \mathbb{C}_\lambda$ is pure of weight zero, and we have a canonical map $\psi : \mathbb{C}_\lambda \rightarrow F_{\geq 0} \mathbb{C}_\lambda$ (from Lemma 2.4). By [BBD82] 5.4.5, we have $F_{\geq 0} \mathbb{C}_\lambda \cong \bigoplus_m {}^p H^m$, where we set ${}^p H^m := {}^p H^m(F_{\geq 0} \mathbb{C}_\lambda)[-m]$. Each ${}^p H^m[m]$ further decomposes into a direct sum of simple perverse sheaves by [BBD82] 5.3.9 i). By Lemma 2.6 4) and the construction of the minimal extension IC_λ of $\underline{\xi}[\dim \mathbb{O}_\lambda]$, we deduce that ${}^p H^m$ contains a direct summand supported outside of $\overline{\mathbb{O}_\lambda} \setminus \mathbb{O}_\lambda$ only if $m = 0$. Moreover, such a direct summand is unique and is isomorphic to IC_λ . It follows that

$$\mathrm{Hom}_G(\mathbb{C}_\lambda, {}^p H^m) \cong \mathrm{Hom}_G(\underline{\xi}[\dim \mathbb{O}_\lambda], j_\lambda^! {}^p H^m) = \begin{cases} \mathbb{C} & (m = 0) \\ \{0\} & (\text{otherwise}) \end{cases}. \quad (2.6)$$

Claim B. *The map ψ is non-zero after projected to a direct factor of $F_{\geq 0} \mathbb{C}_\lambda$.*

Proof. Consider the long exact sequence of perverse cohomologies associated to the distinguished triangles $(F_{<0}\mathbb{C}_\lambda, \mathbb{C}_\lambda, F_{\geq 0}\mathbb{C}_\lambda)$:

$$\cdots \rightarrow {}^p H^i(\mathbb{C}_\lambda) \xrightarrow{{}^p H^i(\psi)} {}^p H^i(F_{\geq 0}\mathbb{C}_\lambda) \rightarrow {}^p H^{i+1}(F_{<0}\mathbb{C}_\lambda) \rightarrow {}^p H^{i+1}(\mathbb{C}_\lambda) \rightarrow \cdots$$

This is an exact sequence in the category of G -equivariant perverse sheaves. We have $[\mathbb{C}_\lambda]_t = [F_{\geq 0}\mathbb{C}_\lambda]_t + [F_{<0}\mathbb{C}_\lambda]_t$ by Lemma 2.4 3). These imply that the connecting homomorphism $\delta_i : {}^p H^i(F_{\geq 0}\mathbb{C}_\lambda) \rightarrow {}^p H^{i+1}(F_{<0}\mathbb{C}_\lambda)$ must be zero for every $i \in \mathbb{Z}$.

If a projection $\phi_{\mathcal{E}}$ of ${}^p H^i(F_{\geq 0}\mathbb{C}_\lambda) \subset F_{\geq 0}\mathbb{C}_\lambda$ ($i \in \mathbb{Z}$) to its (pure) direct factor \mathcal{E} satisfies $\phi_{\mathcal{E}} \circ \psi = 0$, then we have necessarily $\{0\} \neq \delta_i(\mathcal{E}) \subset {}^p H^{i+1}(F_{<0}\mathbb{C}_\lambda)$. This is a contradiction and we conclude that $\phi_{\mathcal{E}} \circ \psi \neq 0$ for every direct factor \mathcal{E} as required. \square

We return to the proof Corollary 2.7. Thanks to (2.6) and Claim B, $F_{\geq 0}\mathbb{C}_\lambda$ has a unique direct factor that is isomorphic to $\mathbb{I}\mathbb{C}_\lambda$. Hence, we conclude $F_{\geq 0}\mathbb{C}_\lambda = \mathbb{I}\mathbb{C}_\lambda$, that is the first part of the first assertion (cf. [K15] Claims A, B). This, together with Lemma 2.6 4), 5), implies that $\text{gr}_a \mathbb{C}_\lambda$ ($a < 0$) is a direct sum of shifted perverse sheaves supported on $\overline{\mathbb{O}_\lambda} \setminus \mathbb{O}_\lambda$. This proves the second assertion, which finishes the proof. \square

In the below (in this section), we additionally assume the setting of section one. In particular, we set $A := A_{(G, \mathfrak{x})}$.

Theorem 2.8 (Standard complexes). *In the setting of Lemma 2.6, we also assume (\spadesuit) in Condition 1.1 and $(\clubsuit)_1$ in Condition 1.4. We normalize \mathcal{L} to be pure of weight zero. Then, we have a complex of graded projective A -modules*

$$(Q(\mathcal{E}), d) : \cdots \rightarrow Q(\mathcal{E})_{a-1} \xrightarrow{d_{a-1}} Q(\mathcal{E})_a \xrightarrow{d_a} Q(\mathcal{E})_{a+1} \xrightarrow{d_{a+1}} \cdots$$

with the following properties:

1. We have $Q(\mathcal{E})_a \cong \text{Ext}_G^\bullet(\text{gr}_a \mathcal{E}, \mathcal{L})$ as a graded A -module with pure weight $-a$;
2. Each differential has degree one and respects the weight structure;
3. We have an isomorphism of graded A -modules:

$$\text{Ext}_G^\bullet(\mathcal{E}, \mathcal{L}) \cong H^\bullet(Q(\mathcal{E}), d). \quad (2.7)$$

Proof. For each $a \in \mathbb{Z}$, the sheaf $\text{gr}_a \mathcal{E}$ is a direct sum of shifted simple (G -equivariant) perverse sheaves by Lemma 2.6 5). In particular, $\text{Ext}_G^\bullet(\text{gr}_a \mathcal{E}, \mathcal{L})$ is a projective A -module. Since A is pure of weight zero (see $(\clubsuit)_1$), each P_λ is pure of some weight (depending on the weight of $\mathbb{I}\mathbb{C}_\lambda$). Therefore, $\text{Ext}_G^\bullet(\text{gr}_a \mathcal{E}, \mathcal{L})$ is pure of weight $-a$ since $\text{gr}_a \mathcal{E}$ is pure of weight a and \mathcal{L} is pure of weight zero.

We define a graded A -module complex

$$(Q_{\geq a}(\mathcal{E}), d) : 0 \rightarrow Q(\mathcal{E})_a \xrightarrow{d_a} Q(\mathcal{E})_{a+1} \xrightarrow{d_{a+1}} Q(\mathcal{E})_{a+2} \rightarrow \cdots$$

with degree one differentials and $Q(\mathcal{E})_b := \text{Ext}_G^\bullet(\text{gr}_b \mathcal{E}, \mathcal{L})$ ($b \geq a$) by induction on $a \in \mathbb{Z}$.

There exists $a_0 \in \mathbb{Z}$ so that $\text{gr}_a \mathcal{E} \cong \{0\}$ for $a \geq a_0$ by Lemma 2.3, and so we define $(Q_{\geq a}(\mathcal{E}), d) := (\{0\}, d)$ for $a \geq a_0$. We assume $(Q_{\geq (a+1)}(\mathcal{E}), d)$ is already defined and the following condition

(\star)_b We have an isomorphism $H^\bullet(Q_{\geq b}(\mathcal{E}), d) \cong \text{Ext}_G^\bullet(F_{\geq b}\mathcal{E}, \mathcal{L})$ as graded A -modules;

for each $b > a$. Note that (\star)_a for $a \ll 0$ is equivalent to the assertion since $F_{\geq a}\mathcal{E} \cong \mathcal{E}$ and $(Q_{\geq a}(\mathcal{E}), d) = (Q(\mathcal{E}), d)$ for $a \ll 0$.

We only need to construct a differential $d_a : Q(\mathcal{E})_a \rightarrow Q(\mathcal{E})_{a+1}$ with degree one in the complex $(Q_{\geq a}(\mathcal{E}), d)$ which satisfies (\star)_a to proceed the induction.

The functor $\text{Ext}_G^\bullet(\bullet, \mathcal{L})$ sends the distinguished triangle (2.5) to the distinguished triangle

$$\rightarrow \text{Ext}_G^\bullet(F_{>a}\mathcal{E}, \mathcal{L}) \rightarrow \text{Ext}_G^\bullet(F_{\geq a}\mathcal{E}, \mathcal{L}) \rightarrow \text{Ext}_G^\bullet(\text{gr}_a \mathcal{E}, \mathcal{L}) \xrightarrow{+1}. \quad (2.8)$$

This is a distinguished triangle of (complexes of) \mathbb{C} -vector spaces. All the maps in (2.8) can be regarded as compositions on the first factor of Ext_G^\bullet . The A -module action on each term on (2.8) is induced from the composition on the second factor of Ext_G^\bullet . Therefore, all the maps in (2.8) must commute with the A -actions by the associativity of compositions (i.e. all the maps are graded A -module homomorphisms). By the induction hypothesis, we deduce that $H^\bullet(Q_{>a}(\mathcal{E}), d) \cong \text{Ext}_G^\bullet(F_{>a}\mathcal{E}, \mathcal{L})$ has weight $< -a$.

We denote the connecting map (the map with degree one) of (2.8) by δ . The graded A -module $\ker \delta$ has pure weight $-a$, and $\text{coker } \delta[-1]$ has weight $< -a$. Since A is pure of weight zero (see $(\clubsuit)_1$), we conclude

$$\begin{aligned} 0 \rightarrow \ker \delta \rightarrow Q(\mathcal{E})_a \xrightarrow{\delta} \text{Ext}_G^\bullet(F_{>a}\mathcal{E}, \mathcal{L}) \langle -1 \rangle \rightarrow \text{coker } \delta \rightarrow 0 \quad (\text{exact}), \quad \text{and} \\ \text{Ext}_G^\bullet(F_{\geq a}\mathcal{E}, \mathcal{L}) \cong \ker \delta \oplus \text{coker } \delta \langle 1 \rangle \quad \text{as graded } A\text{-modules.} \end{aligned} \quad (2.9)$$

By (\star)_{a+1}, the graded A -module $\ker d_{a+1}$ is a direct factor of $H^\bullet(Q_{>a}(\mathcal{E}), d)$, and it is the weight $-(a+1)$ -part of $H^\bullet(Q_{>a}(\mathcal{E}), d)$. Since $Q(\mathcal{E})_a$ have weight $-a$ and δ lowers the weight by one, we conclude

$$\delta(Q(\mathcal{E})_a) \subset \ker d_{a+1} \subset H^\bullet(Q_{>a}(\mathcal{E}), d) \cong \text{Ext}_G^\bullet(F_{>a}\mathcal{E}, \mathcal{L}).$$

It follows that the map δ lifts to a graded map $\delta' : Q(\mathcal{E})_a \rightarrow Q(\mathcal{E})_{a+1} \langle -1 \rangle$. If we set $d_a := \delta'$, then we have

$$\begin{aligned} H^b(Q_{\geq a}(\mathcal{E}), d) &\cong H^b(Q_{\geq a+1}(\mathcal{E}), d) \quad \text{for every } b \geq a+2; \\ H^{a+1}(Q_{\geq a}(\mathcal{E}), d) &\cong \ker d_{a+1} / \delta(Q(\mathcal{E})_a) \subset \text{coker } \delta \langle 1 \rangle; \\ \bigoplus_{b>a} H^b(Q_{\geq a}(\mathcal{E}), d) &\cong \text{coker } \delta \langle 1 \rangle, \quad \text{and} \quad H^a(Q_{\geq a}(\mathcal{E}), d) \cong \ker \delta. \end{aligned} \quad (2.10)$$

By the comparison of (2.10) and (2.9), we deduce that the complex $(Q_{\geq a}(\mathcal{E}), d)$ satisfies our requirement, and hence the induction proceeds as required. \square

Proposition 2.9. Assume (\spadesuit) and $(\clubsuit)_1$. Fix $\lambda = (\underline{\lambda}, \xi) \in \Lambda$. Let us normalize $\underline{\xi}[\dim \mathbb{O}_\lambda]$ to be weight zero. If each IC_γ (whose weight is normalized to be zero) is pointwise pure of weight zero along \mathbb{O}_λ , then $(Q(\mathbb{C}_\lambda), d)$ gives a (finite length) graded projective A -resolution of \tilde{K}_λ .

Proof. By Theorem 2.8, we have $\tilde{K}_\lambda \cong \bigoplus_{a \in \mathbb{Z}} H^a(Q(\mathbb{C}_\lambda), d)$ as graded A -modules. In addition, the direct factor $H^a(Q(\mathbb{C}_\lambda), d) \subset \tilde{K}_\lambda$ has weight $-a$.

By assumption, the graded A -module K_μ is pure of weight zero for every $\mu \sim \lambda$. Hence, so is \tilde{K}_λ by (1.6) since G_λ is affine (note that $\{0\} \neq L_\lambda = \text{Hom}_G(\mathbb{C}_\lambda, L_\lambda \boxtimes \mathbb{C}_\lambda) \subset \tilde{K}_\lambda$ belongs to the weight zero part). As a consequence, $H^\bullet(Q(\mathbb{C}_\lambda), d)$ is pure of weight zero, meaning that $H^i(Q(\mathbb{C}_\lambda), d) = \{0\}$ for every $i \neq 0$. By Corollary 2.7 2), we have $Q(\mathbb{C}_\lambda)_a = \{0\}$ for $a > 0$. Therefore, the standard complex of graded A -modules

$$\cdots \longrightarrow Q(\mathbb{C}_\lambda)_{-3} \xrightarrow{d_{-3}} Q(\mathbb{C}_\lambda)_{-2} \xrightarrow{d_{-2}} Q(\mathbb{C}_\lambda)_{-1} \xrightarrow{d_{-1}} Q(\mathbb{C}_\lambda)_0 \rightarrow \tilde{K}_\lambda \rightarrow 0$$

gives a projective resolution of \tilde{K}_λ . It is of finite length by Lemma 2.6 2). \square

Corollary 2.10. *Assume (\spadesuit) and (\clubsuit) . For each $\lambda \in \Lambda$, we have:*

1. $\tilde{K}_\lambda \in A\text{-gmod}^{pf}$;
2. \tilde{K}_λ is a quotient of P_λ ;
3. The complex $(Q(\mathbb{C}_\lambda), d)$ is a minimal projective resolution of \tilde{K}_λ ;
4. We have $Q(\mathbb{C}_\lambda)_0 \cong P_\lambda$, and $\bigoplus_{a < 0} Q(\mathbb{C}_\lambda)_a$ is a direct sum of one copy of P_λ and finitely many copies of $\{P_\mu \langle k \rangle\}_{\mu \prec \lambda, k \in \mathbb{Z}}$.

Proof. The first assertion follows from Proposition 2.9 and $(\clubsuit)_2$ in Condition 1.4. By Corollary 2.7 1), we deduce that \tilde{K}_λ is a quotient of P_λ , that is the second assertion.

The third and the fourth assertions except for the minimality of the resolution follows by (additionally) using Theorem 2.8 and Corollary 2.7 2).

As we have a finite length finitely generated projective resolution, there exists a minimal projective resolution of \tilde{K}_λ . The complex $(Q(\mathbb{C}_\lambda), d)$ is not a minimal resolution of \tilde{K}_λ if and only if there is a graded indecomposable projective A -direct factor $P \subset Q(\mathbb{C}_\lambda)_a$ ($a \leq 0$) so that the quotient map $\psi : P \rightarrow \text{hd } P$ represents zero in $\text{ext}_A^{-a}(\tilde{K}_\lambda, \text{hd } P)$. For this, we need the following condition $(\dagger)_a$ for some $a \leq 0$:

- $(\dagger)_a$ $\text{Im } d_a \subset Q(\mathbb{C}_\lambda)_{a+1}$ gives a non-zero module in $\text{hd } Q(\mathbb{C}_\lambda)_{a+1}$. In other words, there exist a pair of indecomposable direct summands $P \subset Q(\mathbb{C}_\lambda)_{a+1}$ and $Q \subset Q(\mathbb{C}_\lambda)_a$ so that $P = d_a(Q)$.

We prove that such direct summands are inexistent to deduce the minimality.

The differential d_a of $(Q(\mathbb{C}_\lambda), d)$ is induced by a morphism $F_{>a}\mathbb{C}_\lambda \rightarrow \text{gr}_a \mathbb{C}_\lambda[1]$ arising from the distinguished triangle $(\text{gr}_a \mathbb{C}_\lambda, F_{\geq a} \mathbb{C}_\lambda, F_{>a} \mathbb{C}_\lambda)$. In fact, we only need its pullback ψ to $\text{gr}_{a+1} \mathbb{C}_\lambda$ though the distinguished triangle $(\text{gr}_{a+1} \mathbb{C}_\lambda, F_{>a} \mathbb{C}_\lambda, F_{>a+1} \mathbb{C}_\lambda)$. Here the both of $\text{gr}_a \mathbb{C}_\lambda$ and $\text{gr}_{a+1} \mathbb{C}_\lambda$ are direct sums of shifted G -equivariant perverse sheaves by Lemma 2.6 5). Taking account into this, $d_a (= \text{Ext}_G^\bullet(\psi, \mathcal{L}))$ induces an isomorphism between indecomposable projective A -direct factors of $Q(\mathbb{C}_\lambda)_a$ and $Q(\mathbb{C}_\lambda)_{a+1}$ only if ψ induces an isomorphism between direct factors of $\text{gr}_a \mathbb{C}_\lambda[1]$ and $\text{gr}_{a+1} \mathbb{C}_\lambda$ that are (shifted) irreducible perverse sheaves.

In addition, ψ splits into direct sums of

$$\psi_j^i : {}^p H^i(\text{gr}_{a+1} \mathbb{C}_\lambda)[-i] \longrightarrow {}^p H^{i+j}(\text{gr}_a \mathbb{C}_\lambda)[1-i-j] \quad \text{and} \quad \psi_j := \bigoplus_i \psi_j^i.$$

The morphism ψ_j in $D_G^b(\mathfrak{X})$ is a morphism of perverse sheaves if and only if $j = 1$ (and it is zero for $j > 1$). Therefore, the condition $(\dagger)_a$ implies $\psi_1 \neq 0$.

Consider the long exact sequence of perverse cohomologies associated to the distinguished triangle $(\mathrm{gr}_a \mathbb{C}_\lambda, F_{\geq a} \mathbb{C}_\lambda, F_{> a} \mathbb{C}_\lambda)$:

$$\cdots \rightarrow {}^p H^i(F_{\geq a} \mathbb{C}_\lambda) \rightarrow {}^p H^i(F_{> a} \mathbb{C}_\lambda) \xrightarrow{\delta_i} {}^p H^{i+1}(\mathrm{gr}_a \mathbb{C}_\lambda) \rightarrow {}^p H^{i+1}(F_{\geq a} \mathbb{C}_\lambda) \rightarrow \cdots.$$

Here all the connecting maps δ_i must be zero by Lemma 2.6 4).

The condition $\psi_1^i \neq 0$ implies that the composition map

$${}^p H^i(\mathrm{gr}_{a+1} \mathbb{C}_\lambda) \longrightarrow {}^p H^i(F_{> a} \mathbb{C}_\lambda) \xrightarrow{\delta_{i-1}} {}^p H^{i+1}(\mathrm{gr}_a \mathbb{C}_\lambda)$$

is non-zero, which cannot happen. Therefore, we deduce $\psi_1 = 0$, and hence the condition $(\dagger)_a$ cannot hold.

This proves that the complex $(Q(\mathbb{C}_\lambda), d)$ is a minimal resolution of \tilde{K}_λ , which completes the proof of Corollary 2.10. \square

Proof of Theorem 1.8. For $\mu \not\preceq \lambda = (\underline{\lambda}, \xi)$, we have $j_\lambda^! \mathbb{I} \mathbb{C}_\mu = \{0\}$ by the support condition. It follows that

$$[\tilde{K}_\lambda : L_\mu \langle i \rangle]_0 = \dim \mathrm{Ext}_{D_G^b(\mathfrak{X})}^i(\mathbb{C}_\lambda, \mathbb{I} \mathbb{C}_\mu) = \dim \mathrm{Ext}_{D_G^b(\mathbb{O}_\lambda)}^{i - \dim \mathbb{O}_\lambda}(\underline{\xi}, j_\lambda^! \mathbb{I} \mathbb{C}_\mu) = 0, \quad (2.11)$$

$$[K_\lambda : L_\mu \langle i \rangle]_0 = \dim H^{i + \dim \mathbb{O}_\lambda} i_\lambda^! \mathbb{I} \mathbb{C}_\mu = 0 \quad (2.12)$$

for every $i \in \mathbb{Z}$.

For each $\lambda = (\underline{\lambda}, \xi), \mu = (\underline{\lambda}, \zeta) \in \Lambda$ (so that $\lambda \sim \mu$), we have

$$[K_\lambda : L_\mu \langle i \rangle]_0 = \dim \mathrm{Hom}_{C_\lambda}(\xi, H^{i + \dim \mathbb{O}_\lambda} i_\lambda^! \mathbb{I} \mathbb{C}_\mu) = \begin{cases} \delta_{\lambda, \mu} & (i = 0) \\ 0 & (i \neq 0) \end{cases}. \quad (2.13)$$

This proves Theorem 1.8 1). In addition, we have

$$[\tilde{K}_\lambda : L_\mu \langle i \rangle]_0 = \dim \mathrm{Ext}_{D_G^b(\mathbb{O}_\lambda)}^{i - \dim \mathbb{O}_\lambda}(\underline{\xi}, j_\lambda^! \mathbb{I} \mathbb{C}_\mu) = \dim \mathrm{Hom}_{C_\lambda}(\xi, \zeta \otimes_{\mathbb{C}} H_{G_\lambda^\circ}^i(\mathrm{pt})). \quad (2.14)$$

This proves the latter half of Theorem 1.8 4).

Corollary 2.10 and (2.11) implies

$$\mathrm{ext}_A^\bullet(\tilde{K}_\lambda, \tilde{K}_\mu) = \{0\} \quad \text{for each } \mu \not\preceq \lambda,$$

that is the first half of Theorem 1.8 2).

Similarly, Corollary 2.10, (2.12), and (2.13) implies

$$\mathrm{ext}_A^\bullet(\tilde{K}_\lambda, L_\mu) \neq \{0\} \quad (2.15) \quad \text{or} \quad \mathrm{ext}_A^\bullet(\tilde{K}_\lambda, K_\mu) \neq \{0\} \quad (2.16)$$

only if $\mu \preceq \lambda$. The equation (2.16) is the latter half of Theorem 1.8 2).

The equations (2.15) and (2.11) imply

$$\mathrm{hom}_A(\tilde{K}_\lambda, L_\mu) = \{0\} \quad \text{for every } \lambda \neq \mu \in \Lambda. \quad (2.17)$$

Thanks to Corollary 2.10 4), we have

$$\sum_{i \geq 0} \dim \operatorname{ext}_A^i(\tilde{K}_\lambda, L_\lambda) \leq 1. \quad (2.18)$$

By Corollary 2.10 2), we have a quotient map $P_\lambda \rightarrow \tilde{K}_\lambda$. Taking (2.11) into account, it induces a surjective morphism

$$\psi_\lambda : P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right) \longrightarrow \tilde{K}_\lambda.$$

Let $\ker := \ker \psi_\lambda$. Each simple quotient of \ker is of the form $L_\gamma \langle i \rangle$ with $\gamma \neq \lambda$ and $i > 0$. If $\ker \neq \{0\}$, then the Yoneda interpretation of ext^1 implies

$$\operatorname{ext}_A^1(\tilde{K}_\lambda, L_\gamma) \neq \{0\},$$

which contradicts with (2.15) and (2.18). Therefore, we conclude that $\ker = \{0\}$. This proves Theorem 1.8 3).

It remains to prove the former part of Theorem 1.8 4). Fix a \sim -equivalence class \mathcal{O} in Λ . Set $C := C_\lambda$ and $H_{G_\lambda^\circ}^\bullet := H_{G_\lambda^\circ}^\bullet(\operatorname{pt})$ for a(ny) choice $\lambda \in \mathcal{O}$.

By the functoriality of the Hom-groups of triangulated categories, we deduce that

$$\operatorname{Ext}_G^\bullet(\mathcal{L}, \mathcal{L}) \rightarrow \operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L}) \quad (2.19)$$

is a morphism of graded algebras.

Consider a graded algebra

$$A_{\mathcal{O}} := H_{G_\lambda^\circ}^\bullet \otimes_{\mathbb{C}} \bigoplus_{\lambda=(\underline{\lambda}, \xi), \mu=(\underline{\lambda}, \zeta) \in \mathcal{O}} \operatorname{hom}_{\mathbb{C}}(\xi \boxtimes K_\lambda, \zeta \boxtimes K_\mu) \quad (2.20)$$

with the diagonal C -action. This algebra has a unique self-dual graded simple module $\bigoplus_{\lambda=(\underline{\lambda}, \xi) \in \mathcal{O}} \xi \boxtimes K_\lambda$ (that can be split-off if we take the C -action into account). Note that \tilde{K}_λ is a direct summand of the RHS of (2.19) by $\operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L}) \cong \operatorname{Ext}_G^\bullet((j_\mu)_! j_\mu^! \mathcal{L}, \mathcal{L})$. In addition, the isomorphism $j_\mu^!(j_\mu)_! j_\mu^! \mathcal{L} \cong j_\mu^! \mathcal{L}$ implies the following commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}_G^\bullet((j_\mu)_! j_\mu^! \mathcal{L}, \mathcal{L}) \times \operatorname{Ext}_G^\bullet(\mathcal{L}, \mathcal{L}) & \longrightarrow & \operatorname{Ext}_G^\bullet((j_\mu)_! j_\mu^! \mathcal{L}, \mathcal{L}) \\ \downarrow & & \downarrow \\ \operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L}) \times \operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L}) & \longrightarrow & \operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L}), \end{array}$$

whose horizontal arrows are the A -action and the composition, and the vertical arrows are induced by the pullbacks, respectively. This implies that the A -action on \tilde{K}_λ factors through the algebra $\operatorname{Ext}_G^\bullet(j_\mu^! \mathcal{L}, j_\mu^! \mathcal{L})$, that is isomorphic to $(A_{\mathcal{O}})^C \subset A_{\mathcal{O}}$.

For each $k \geq 0$, the ideal $H_{G_\lambda^\circ}^{\geq k} \subset H_{G_\lambda^\circ}^\bullet$ generates a two-sided ideal $J^k \subset A_{\mathcal{O}}$. Here J^1 is the graded Jacobson radical of $A_{\mathcal{O}}$. Consider the graded $A_{\mathcal{O}}$ -module

$$\tilde{K} := H_{G_\lambda^\circ}^\bullet \otimes_{\mathbb{C}} \bigoplus_{\mu=(\underline{\lambda}, \zeta) \in \mathcal{O}} \zeta \boxtimes K_\mu. \quad (2.21)$$

We have a natural identification $\tilde{K} = \bigoplus_{\lambda=(\underline{\lambda}, \xi) \in \mathcal{O}} \xi \boxtimes \tilde{K}_\lambda$ through the algebra map $A \rightarrow (A_{\mathcal{O}})^C$. For each k ,

$$J^k \tilde{K} / J^{k+1} \tilde{K} \cong H_{G_\lambda^\circ}^k \otimes_{\mathbb{C}} \bigoplus_{\mu=(\underline{\lambda}, \zeta) \in \mathcal{O}} \zeta \boxtimes K_\lambda$$

is a semisimple graded $A_{\mathcal{O}}$ -module. This induces a filtration of \tilde{K}_λ whose associated graded is a direct sum of the grading shifts of K_μ with $\mu \in \mathcal{O}$, which completes the proof of Theorem 1.8. \square

3 Applications of Theorem 1.8

Keep the setting of section one. We assume the conditions (\spadesuit) in Condition 1.1, (\clubsuit) in Condition 1.4, and work over an algebraic closure of a finite field unless stated otherwise.

Lemma 3.1. *For each $\lambda \in \Lambda$, we have $\dim K_\lambda < \infty$.*

Proof. The assertion is equivalent to $\dim H^\bullet i_\lambda^! \mathcal{L} < \infty$, which follows from the fact that \mathcal{L} is a constructible sheaf. \square

Corollary 3.2. *For each $\lambda \in \Lambda$, the A -module K_λ has a finite composition series.* \square

Proposition 3.3. *We assume (\spadesuit) and (\clubsuit) . For each $\lambda \in \Lambda$, K_λ admits a finite resolution by the grading shifts of $\{\tilde{K}_\gamma\}_{\gamma \sim \lambda}$. In addition, we have*

$$\mathrm{ext}_A^\bullet(K_\lambda, K_\mu) = \{0\} = \mathrm{ext}_A^\bullet(K_\lambda, L_\mu) \text{ for each } \mu \not\sim \lambda.$$

Proof. Fix a \sim -equivalence class \mathcal{O} in Λ , and we borrow the notation (namely $A_{\mathcal{O}}, C, \tilde{K}$, and $H_{G_\lambda^\circ}^\bullet$) and setting from the proof of the former half of Theorem 1.8 4). Here we set $A'_\mathcal{O} := (A_{\mathcal{O}})^C$ (cf. (2.20)) for simplicity during this proof.

For each $\lambda \in \mathcal{O}$, \tilde{K}_λ is a projective object in the full subcategory $A\text{-gmod}_\mathcal{O}$ of $A\text{-gmod}$ consisting of objects M such that $[M : L_\gamma] = 0$ if $\gamma \not\sim \lambda \in \mathcal{O}$ by Theorem 1.8 3).

In particular, we have

$$\mathrm{gdim} \mathrm{hom}_A(\tilde{K}_\lambda, \tilde{K}_\mu) = \mathrm{gdim} \mathrm{Hom}_C(\xi, \zeta \otimes_{\mathbb{C}} H_{G_\lambda^\circ}^\bullet)$$

for each $\lambda = (\underline{\lambda}, \xi), \mu = (\underline{\lambda}, \zeta) \in \mathcal{O}$.

Consider the graded $A'_\mathcal{O}$ -module

$$M_\lambda := \mathrm{hom}_C(\xi, H_{G_\lambda^\circ}^\bullet \otimes_{\mathbb{C}} \bigoplus_{\mu=(\underline{\lambda}, \zeta) \in \mathcal{O}} \zeta \boxtimes K_\mu)$$

for each $\lambda = (\underline{\lambda}, \xi) \in \mathcal{O}$, that is the ξ -isotypic component of \tilde{K} (cf. (2.20) and (2.21)).

The factorization of the A -action on \tilde{K}_λ ($\lambda \in \mathcal{O}$) through $A'_\mathcal{O}$ identifies M_λ with \tilde{K}_λ . This also identifies (grading shifts of) K_λ (that are subquotients of \tilde{K}_λ) with a simple graded $A'_\mathcal{O}$ -module.

For each $\lambda \in \mathcal{O}$, we deduce that M_λ is a projective $A'_\mathcal{O}$ -module with a simple head K_λ by replacing L_λ with K_λ and (G, \mathfrak{X}) with (G_λ, pt) in the discussion of section one. It follows that

$$\text{gdim hom}_{A'_\mathcal{O}}(M_\lambda, M_\mu) = \text{gdim hom}_C(\xi, \zeta \otimes_C H_{G_\lambda}^\bullet)$$

for each $\lambda = (\underline{\lambda}, \xi), \mu = (\underline{\lambda}, \zeta) \in \mathcal{O}$.

The map $\psi \in \text{end}_{A'_\mathcal{O}}(M_\lambda, M_\mu)$ is uniquely determined by choosing the image of K_λ , and we have a canonical quotient $K_\lambda \rightarrow L_\lambda$ (or its kernel) singled out by its restriction to the image of $A \rightarrow A'_\mathcal{O}$. Thus, we have a natural inclusion

$$\text{hom}_{A'_\mathcal{O}}(M_\lambda, M_\mu) \hookrightarrow \text{hom}_A(\tilde{K}_\lambda, \tilde{K}_\mu),$$

that is in fact an isomorphism by the comparison of graded dimensions.

This shows that

$$\text{end}_{A'_\mathcal{O}}\left(\bigoplus_{\lambda \in \mathcal{O}} M_\lambda\right) \cong \text{end}_A\left(\bigoplus_{\lambda \in \mathcal{O}} \tilde{K}_\lambda\right) \quad (3.1)$$

as graded algebras. Here the LHS must be Morita equivalent to $A'_\mathcal{O}$.

We set $R_C := \mathbb{C}C \ltimes H_{G_\lambda}^\bullet$, and regard it as a graded algebra by setting $\deg C = 0$ and $\deg H_{G_\lambda}^k = k$ for each $k \geq 0$. Each $\xi \in \text{Irr } C$ gives rise to an indecomposable graded R_C -module $P_\xi^C := H_{G_\lambda}^\bullet \otimes_C \xi$, and we have $R_C \cong \bigoplus_{\xi \in \text{Irr } C} P_\xi^C \boxtimes \xi$ as graded left R_C -modules (cf. [K15] 1.2). (The natural presentation of R_C is $\bigoplus_{\xi \in \text{Irr } C} P_\xi^C \boxtimes \xi^\vee$. Here we employ the current notation for the simplicity of formulas.)

If we further replace each M_λ for $\lambda = (\underline{\lambda}, \xi) \in \Lambda$ with $M_\lambda \boxtimes \xi$ in (3.1), then we find an isomorphism

$$R_C = \text{end}_{R_C}\left(\bigoplus_{\lambda=(\underline{\lambda}, \xi) \in \mathcal{O}} P_\xi^C \boxtimes \xi\right) \xrightarrow{\psi} \text{end}_{A'_\mathcal{O}}\left(\bigoplus_{\lambda=(\underline{\lambda}, \xi) \in \mathcal{O}} M_\lambda \boxtimes \xi\right).$$

Here we have $\psi = \bigoplus_{\kappa \in \text{Irr } C} \psi_\kappa$, where ψ_κ sends $\zeta^\vee \boxtimes \zeta \ltimes \theta \in \mathbb{C}C \ltimes H_{G_\lambda}^\bullet$ to

$$\text{hom}_\mathbb{C}(K_\mu \boxtimes \zeta, K_\mu \otimes_C \text{hom}_C(\kappa, \zeta \otimes_C \theta) \boxtimes \kappa) \subset \text{hom}_\mathbb{C}(M_\mu \boxtimes \zeta, M_\gamma \boxtimes \kappa),$$

for each $\mu = (\underline{\lambda}, \zeta), \gamma = (\underline{\lambda}, \kappa) \in \mathcal{O}$ and an irreducible C -submodule $\theta \subset H_{G_\lambda}^\bullet$. This shows that R_C is Morita equivalent to $A'_\mathcal{O}$.

Here (3.1) exhibits that every graded A -module morphism between grading shifts of $\{\tilde{K}_\lambda\}_{\lambda \in \mathcal{O}}$ is in fact a pullback of a graded $A'_\mathcal{O}$ -module morphism. In particular, its kernel and cokernel admits a decreasing separable $\{K_\lambda \langle j \rangle\}_{\lambda \in \mathcal{O}, j \in \mathbb{Z}}$ -filtration. Hence, so is the homology of a graded complex whose terms are direct sums of grading shifts of $\{\tilde{K}_\lambda\}_{\lambda \in \mathcal{O}}$.

By [MR01] 7.5.6, the global dimension of the algebra R_C is the same as that of $H_{G_\lambda}^\bullet$, that is a polynomial ring. In particular, every simple module of R_C admits a finite length projective resolution.

As a consequence, each K_λ admits a finite length graded projective resolution as a graded $A'_\mathcal{O}$ -module, and hence also admits a finite length graded $\{\tilde{K}_\lambda \langle j \rangle\}_{\lambda \in \mathcal{O}, j \in \mathbb{Z}}$ -resolution as a graded A -module. This is the first assertion.

For each $\lambda \in \mathcal{O}$, we replace K_λ by its finite resolution whose terms are grading shifts of $\{\tilde{K}_\gamma\}_{\gamma \in \mathcal{O}}$ to compute $\text{ext}_A^\bullet(K_\lambda, K_\mu)$ and $\text{ext}_A^\bullet(K_\lambda, L_\mu)$ for $\mu \not\sim \lambda$ via double complexes. Since we know

$$\text{ext}_A^\bullet(\tilde{K}_\gamma, K_\mu) = \{0\} = \text{ext}_A^\bullet(\tilde{K}_\gamma, L_\mu)$$

for $\gamma \sim \lambda$ by Theorem 1.8 2) and (2.15) in the proof of Theorem 1.8, the double complexes must be entirely zero. This shows the second assertion. \square

Corollary 3.4. *For each $\lambda \in \Lambda$, the A -module L_λ admits a (graded) projective resolution of finite length.*

Proof. Combining Proposition 3.3 and Corollary 2.10 (through a double complex), we deduce that each K_λ admits a finite length finitely generated graded projective resolution. By Corollary 3.2 and Theorem 1.8 1), we see that each L_λ is constructed by a finitely many successive short exact sequences from $\{K_\mu \langle i \rangle\}_{\mu \in \Lambda, i \in \mathbb{Z}}$. Thus, we conclude

$$\sum_{\mu \in \Lambda} \dim \text{ext}_A^\bullet(L_\lambda, L_\mu) < \infty.$$

Therefore, a minimal projective resolution of L_λ (that exists by Corollary 1.13) contains finitely many indecomposable projective A -modules as its direct summands (counted with multiplicities), which proves the assertion. \square

Theorem 3.5. *Assume the conditions (\spadesuit) and (\clubsuit) . Then, the algebra A has finite global dimension. In particular, we have $A\text{-gmod}^{pf} \xrightarrow{\cong} A\text{-gmod}$.*

Proof. The graded algebra A is (left and right) Noether, and is graded Morita equivalent to B . The graded algebra B is non-negatively graded and B^0 is (canonically isomorphic to) the semi-simple quotient of the graded Jacobson radical $B^{>0}$ of B . Therefore, we apply Li's result [Li96] (cf. [NO82] D.VII) to Corollary 3.4 to conclude the result. \square

Remark 3.6. Applying the arguments of section two by an induction with respect to the closure relation (from open orbits), we can use the condition of Theorem 1.6 a) instead $(\clubsuit)_2$ to deduce Theorem 3.5.

For each $M \in A\text{-gmod}$, we define its graded character as:

$$\text{gch } M := \sum_{\lambda \in \Lambda} [M : L_\lambda] [L_\lambda] \in \bigoplus_{\lambda \in \Lambda} \mathbb{Q}((t)) [L_\lambda].$$

Lemma 3.7. *Each of the collections $\{\text{gch } K_\lambda\}_{\lambda \in \Lambda}$ and $\{\text{gch } \tilde{K}_\lambda\}_{\lambda \in \Lambda}$ is a $\mathbb{Q}((t))$ -basis of $\bigoplus_{\lambda \in \Lambda} \mathbb{Q}((t)) [L_\lambda]$.*

Proof. This is a rephrasing of Corollary 1.10. \square

For each $M \in A\text{-gmod}$, we have collections of elements $[M : P_\lambda]$, $[M : K_\lambda]$, and $[M : \tilde{K}_\lambda]$ in $\mathbb{Q}((t))$ so that

$$\text{gch } M = \sum_{\lambda \in \Lambda} [M : P_\lambda] \text{gch } P_\lambda = \sum_{\lambda \in \Lambda} [M : K_\lambda] \text{gch } K_\lambda = \sum_{\lambda \in \Lambda} [M : \tilde{K}_\lambda] \text{gch } \tilde{K}_\lambda.$$

Thanks to Theorem 3.5, every module in $A\text{-gmod}$ can be a (first) variable of the graded Euler-Poincaré pairing (1.7), and $[M : P_\lambda] \in \mathbb{Z}[t^{\pm 1}]$.

Lemma 3.8. *If $M \in A\text{-gmod}$ is finite-dimensional, then its graded dual M^* belongs to $A\text{-gmod}$. In addition, we have*

$$[M : L_\mu] = \overline{[M^* : L_{\mu^\vee}]}.$$

Proof. See Lemma 1.2 and [K15] Lemma 2.5. \square

Proposition 3.9. *Assume the conditions (\spadesuit) and (\clubsuit) . We have*

$$\text{ext}_A^i(\tilde{K}_\lambda, K_{\mu^\vee}^*) \cong \begin{cases} \mathbb{C} & (\lambda = \mu, i = 0) \\ \{0\} & (\text{otherwise}) \end{cases}, \text{ and } \langle \tilde{K}_\lambda, K_{\mu^\vee}^* \rangle_{\text{gEP}} = \begin{cases} 1 & (\lambda = \mu) \\ 0 & (\lambda \neq \mu) \end{cases}.$$

Remark 3.10. Proposition 3.9 can be seen as the “dual picture” of Mirollon-Vilonen and Beilinson-Ginzburg-Soergel [MV87, BGS96], or the equivariant picture of Chriss-Ginzburg [CG97] §8.7.

Proof of Proposition 3.9. Since the second assertion follows from the first one, we prove only the first assertion.

By Corollary 2.10 4), Lemma 3.8, and Theorem 1.8 1), we deduce the case $\mu \not\prec \lambda$. In this case, we further deduce

$$\text{ext}_A^\bullet(K_\lambda, K_{\mu^\vee}^*) \cong \{0\} \quad \text{if } \lambda \not\prec \mu \quad (3.2)$$

by Proposition 3.3.

Thus, we assume $\mu \prec \lambda$ in the below. Consider the derived functors of

$$\text{hom}_A(M, N) \cong \text{hom}_A(N^*, M^*).$$

Since $*$ is an exact functor (from $A\text{-gmod}$ to its dual category) and ext_A^* is an universal δ -functor on them, we conclude that both of them define mutually isomorphic derived functors. In particular, (3.2) also holds for $\mu \prec \lambda$.

As the grading of A is bounded from the below, so are the gradings of K_γ for every $\gamma \in \Lambda$ (say $A^i = \{0\}$ and $K_\gamma^i = \{0\}$ for every $\gamma \in \Lambda$ and $i < c$). Hence, we have $(K_{\mu^\vee}^*)^i = \{0\}$ for every $i > -c$. Let ℓ be the global dimension of A , that is finite by Theorem 3.5. For each $j \in \mathbb{Z}$, there exists a member $K_j^\perp \subset \tilde{K}_\lambda$ of a decreasing separable A -module filtration in Theorem 1.8 4) such that **a)** $(K_j^\perp)^i = \{0\}$ for $i \leq -j - (\ell + 2)c$, and **b)** $\tilde{K}_\lambda/K_j^\perp$ is a finite successive self-extension of (grading shifts) of K_γ with $\gamma \sim \lambda$. Then, the minimal projective resolution of K_j^\perp is concentrated in degree $i > -j - c$ by Corollary 1.13. In particular, (3.2) implies

$$\text{ext}_A^\bullet(K_j^\perp, K_{\mu^\vee}^*)^j = \{0\} = \text{ext}_{R_\beta}^\bullet(\tilde{K}_\lambda/K_j^\perp, K_{\mu^\vee}^*).$$

This yields $\text{ext}_A^\bullet(\tilde{K}_\lambda, K_{\mu^\vee}^*)^j = \{0\}$ (for each j) as required. \square

Corollary 3.11. *Keep the setting of Proposition 3.9. We have*

$$\text{ext}_A^\bullet(K_\lambda, K_\mu^*) = \{0\} \quad \text{and} \quad \langle K_\lambda, K_\mu^* \rangle_{\text{gEP}} = 0 \quad (\lambda \not\prec \mu).$$

Proof. See (3.2) and the third paragraph of the proof of Proposition 3.9. \square

For each $X, Y \in \{P, \tilde{K}, K, L\}$, we define a $\mathbb{Q}((t))$ -valued $\#\Lambda$ -square matrix as $[X : Y] = ([X_\lambda : Y_\mu])_{\lambda, \mu \in \Lambda}$. In particular, we have $[P : \tilde{K}] = ([P_\lambda : \tilde{K}_\mu])_{\lambda, \mu \in \Lambda}$ for $X = P, Y = L$. Similarly, we have $[P : \tilde{K}], [\tilde{K} : K]$, etc...

Corollary 3.12 (Brauer-Humphreys type reciprocity). *Assume the conditions (\spadesuit) and (\clubsuit) . Then, we have*

$$[P_\lambda : \tilde{K}_\mu] = [K_\mu : L_\lambda] \quad \text{for each } \lambda, \mu \in \Lambda.$$

In addition, we have the matrix identity

$$[P : L] = [P : \tilde{K}][\tilde{K} : K][K : L] = {}^t[K : L][\tilde{K} : K][K : L]. \quad (3.3)$$

Proof. We have

$$\begin{aligned} \delta_{\lambda, \mu} &= \langle P_\lambda, L_\mu \rangle_{\text{gEP}} = \langle P_\lambda, L_{\mu^\vee}^* \rangle_{\text{gEP}} = \sum_{\gamma, \delta} \overline{[P_\lambda : \tilde{K}_\gamma]} \overline{[L_\mu : K_\delta]} \langle \tilde{K}_\gamma, K_{\delta^\vee}^* \rangle_{\text{gEP}} \\ &= \sum_{\gamma} \overline{[P_\lambda : \tilde{K}_\gamma]} \overline{[L_\mu : K_\gamma]} \end{aligned}$$

by Proposition 3.9. By applying the bar involution, this shows

$$([P_\lambda : \tilde{K}_\gamma])([K_\delta : L_\mu])^{-1} = (\delta_{\lambda, \mu}),$$

which is equivalent to the first assertion. The first equality of (3.3) is the definition. The second equality of (3.3) follows by $[P : \tilde{K}] = {}^t[K : L]$, that is the first assertion. \square

Corollary 3.13 (Cartan determinant formula). *Assume that $C_\lambda = \{1\}$ for every $\lambda \in \Lambda$. We have*

$$\det [P : L] = \prod_{\lambda \in \Lambda} \text{gdim } H_{G_\lambda}^\bullet(\text{pt}).$$

Remark 3.14. **1)** It is straight-forward to formulate an analogue of Corollary 3.13 without the extra assumption $C_\lambda = \{1\}$ for every $\lambda \in \Lambda$. We choose the current formulation for the sake of simplicity; **2)** Corollary 3.13 asserts $\det [P : L] \neq 0$. Hence the knowledge of Λ , \prec , and $[P : L]$ are enough to determine the matrix $(\text{gdim } H^{\bullet + \dim \mathbb{O}_\lambda} i_\lambda^! | \mathbb{C}_\mu)_{\lambda, \mu} = ([K_\lambda : L_\mu])_{\lambda, \mu}$ by Theorem 1.8 1), 4) and (3.3) (that is a version of the Lusztig-Shoji algorithm; cf. [K15] and Remark 5.9 1)).

Proof of Corollary 3.13. We have $\det [P : L] = (\det [K : L])^2 \det [\tilde{K} : K]$ by (3.3). By Theorem 1.8 1), we have $\det [K : L] = 1$. Hence, the result follows from Theorem 1.8 4). \square

4 An inheritance property of (\spadesuit) and (\clubsuit)

Keep the setting of the previous section. In particular, we assume the conditions (\spadesuit) in Condition 1.1, and (\clubsuit) in Condition 1.4 (for the pair (G, \mathfrak{X})), and work over an algebraic closure of a finite field unless stated otherwise.

For $\underline{\lambda} \in \underline{\Lambda}$, we set $e_{\underline{\lambda}} := \sum_{\mu=(\underline{\lambda}, \xi) \in \Lambda} e_\mu$. We denote by \prec the partial order on $\underline{\Lambda}$ induced from \prec on Λ by simplicity.

The following proof of Theorem 4.1 (modulo Theorem 3.12) is rather well-understood (see e.g. Donkin [Don98]).

Theorem 4.1. *Assume the conditions (\spadesuit) and (\clubsuit) . Then, each P_λ admits a decreasing A -module filtration so that its associated graded is a finite direct sum of A -modules of the form $\tilde{K}_\mu \langle i \rangle$ with $\mu \leq \lambda$ and $i \geq 0$.*

Proof. Let us introduce a total order $<$ on $\underline{\Lambda}$ which refines \prec . We name elements of $\underline{\Lambda}$ as $\{\mathcal{O}_1, \mathcal{O}_2, \dots\}$ so that $\mathcal{O}_i < \mathcal{O}_j$ if $i < j$. We set $e_i := \sum_{j < i} e_{\mathcal{O}_j}$. We also identify \mathcal{O}_j with its preimage in Λ by abuse of notation.

For each $1 \leq i, j \leq \#\Lambda$ and $\lambda \in \mathcal{O}_i$, we put $P_\lambda^{(j)} := P_\lambda / Ae_j P_\lambda$. We have $P_\lambda^{(i)} \cong \tilde{K}_\lambda$ by Theorem 1.8 1) and 3). Since $P_\lambda^{(j)}$ is the quotient of P_λ obtained by annihilating L_μ with $\mu \in \mathcal{O}_k$ for $k < j$, the Yoneda interpretation of ext^1 yields

$$\text{ext}_A^1(P_\lambda^{(j)}, L_\mu) = \{0\} \quad \text{for every } \mu \in \mathcal{O}_k \text{ with } j \leq k. \quad (4.1)$$

Applying long exact sequences repeatedly, we deduce

$$\text{ext}_A^1(P_\lambda^{(j)}, K_\mu^*) = \{0\} \quad \text{for every } \mu \in \mathcal{O}_k \text{ with } j \leq k. \quad (4.2)$$

since we have $[K_\mu^* : L_\nu] \neq 0$ only if $\nu \in \mathcal{O}_l$ with $k \leq l$ by Theorem 1.8 1) and Lemma 3.8. We have a short exact sequence

$$0 \rightarrow \ker_\lambda^j \rightarrow P_\lambda^{(j)} \rightarrow P_\lambda^{(j+1)} \rightarrow 0 \quad (4.3)$$

for each $1 \leq i, j \leq \#\Lambda$ and $\lambda \in \mathcal{O}_i$. By a comparison of multiplicities, we deduce

$$0 = [P_\lambda^{(j)} : L_\mu] = [\ker_\lambda^j : L_\mu] \quad \text{if } \mu \in \mathcal{O}_k \text{ with } j > k. \quad (4.4)$$

This, together with Theorem 1.8 3) and the construction, implies that each \ker_λ^j is a quotient of a direct sum of grading shifts of \tilde{K}_μ with $\mu \in \mathcal{O}_j$.

To prove the claim, it suffices to show that \ker_λ^j is a direct sum of grading shifts of \tilde{K}_μ with $\mu \in \mathcal{O}_j$ for each $j \leq i$ by a downward induction on j . The case $j = i$ is clear as $P_\lambda^{(i)} = \ker_\lambda^i = \tilde{K}_\lambda$. We assume that $\ker_\lambda^{j'}$ is a direct sum of grading shifts of \tilde{K}_μ with $\mu \in \mathcal{O}_{j'}$ for each $j < j' \leq i$ to deduce that \ker_λ^j is a direct sum of grading shifts of \tilde{K}_μ with $\mu \in \mathcal{O}_j$. For each $j \leq k$ and $\mu \in \mathcal{O}_k$, (4.3) yields an exact sequence

$$\text{ext}_A^1(P_\lambda^{(j+1)}, K_\mu^*) \rightarrow \text{ext}_A^1(P_\lambda^{(j)}, K_\mu^*) \rightarrow \text{ext}_A^1(\ker_\lambda^j, K_\mu^*) \rightarrow \text{ext}_A^2(P_\lambda^{(j+1)}, K_\mu^*). \quad (4.5)$$

By induction hypothesis and Proposition 3.9, the most LHS/RHS of (4.5) are $\{0\}$. Hence, applying (4.2) yields

$$\text{ext}_A^1(\ker_\lambda^j, K_\mu^*) = \{0\} \quad \text{for every } \mu \in \mathcal{O}_k \text{ with } j \leq k. \quad (4.6)$$

Let K be the minimal direct sum of (grading shifts of) $\{\tilde{K}_\mu\}_{\mu \in \mathcal{O}_j}$ which surjects onto \ker_λ^j . Let $K' := \ker(K \rightarrow \ker_\lambda^j)$. By the Noetherian hypothesis of A , we deduce that \ker_λ^j is finitely generated. Thus, K and K' are also finitely generated. For each $j \leq k$ and $\mu \in \mathcal{O}_k$, we apply $\text{ext}_A^\bullet(\bullet, K_\mu^*)$ to deduce

$$0 \rightarrow \text{hom}_A(\ker_\lambda^j, K_\mu^*) \xrightarrow{g} \text{hom}_A(K, K_\mu^*) \rightarrow \text{hom}_A(K', K_\mu^*) \rightarrow \text{ext}_A^1(\ker_\lambda^j, K_\mu^*).$$

Since the image of a non-zero map $\tilde{K}_\gamma \rightarrow K_{\gamma^\vee}^*$ is L_γ for each $\gamma \in \Lambda$ (cf. Proposition 3.9), the map g is surjective by the construction of K . Hence, (4.6) implies

$$\text{hom}_A(K', K_\mu^*) = \{0\} \quad \text{for every } \mu \in \mathcal{O}_k \text{ with } j \leq k.$$

We have $[K : L_\nu] \neq \{0\}$ only if $\nu \in \mathcal{O}_k$ with $j \leq k$ by Theorem 1.8 1). Hence, the same is true for $[K' : L_\nu]$. It follows that $K' = \{0\}$ by Corollary 1.12. Therefore, $K \cong \ker_\lambda^j$ is isomorphic to a finite direct sum of (grading shifts of) \tilde{K}_μ with $\mu \in \mathcal{O}_j$. This proceeds the induction and we conclude the result. \square

Corollary 4.2. *Let $\lambda = (\underline{\lambda}, \xi) \in \Lambda$ be a label of a closed G -orbit of \mathfrak{X} . Then, for each $\lambda \not\sim \gamma \in \Lambda$, we have*

$$\mathrm{ext}_A^*(A, L_\gamma) \cong \mathrm{ext}_A^*(A/(Ae_{\underline{\lambda}}A), L_\gamma).$$

Proof. By the construction in the proof of Theorem 4.1, we know that $Ae_{\underline{\lambda}}A$ is isomorphic to a direct sum of grading shifts of $\tilde{K}_\mu \cong P_\mu$ with $\mu \sim \lambda$. It follows that

$$0 \rightarrow Ae_{\underline{\lambda}}A \rightarrow A \rightarrow A/(Ae_{\underline{\lambda}}A) \rightarrow 0 \quad (\text{exact})$$

is a projective resolution with $\mathrm{hom}_A(Ae_{\underline{\lambda}}A, L_\gamma) = \{0\}$, which proves the assertion. \square

Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G -stable subvariety. We form a graded algebra

$$A_{(G, \mathfrak{Y})} := \mathrm{Ext}_G^\bullet(j^*\mathcal{L}, j^*\mathcal{L}).$$

For each $\lambda \in \Lambda$ so that $\mathbb{O}_\lambda \subset \mathfrak{Y}$, the module \tilde{K}_λ is in common between $A_{(G, \mathfrak{X})}$ and $A_{(G, \mathfrak{Y})}$ by construction.

Proposition 4.3. *Let $i : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$ be the inclusion of a closed G -orbit (with $\lambda = (\underline{\lambda}, \xi) \in \Lambda$), and let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be its complement. Then, we have an algebra map $\varpi : A_{(G, \mathfrak{X})} \rightarrow A_{(G, \mathfrak{Y})}$ which induces an isomorphism*

$$A_{(G, \mathfrak{X})}/(A_{(G, \mathfrak{X})}e_{\underline{\lambda}}A_{(G, \mathfrak{X})}) \xrightarrow{\cong} A_{(G, \mathfrak{Y})}.$$

In addition, $\ker \varpi$ is a direct sum of $\{\tilde{K}_\mu \langle k \rangle\}_{\mu \sim \lambda, k \in \mathbb{Z}}$.

Proof of Proposition 4.3. We set $A' := A_{(G, \mathfrak{Y})}$ and $A := A_{(G, \mathfrak{X})}$. The algebra map $\varpi : A \rightarrow A'$ is the restriction map

$$\mathrm{Ext}_G^*(\mathcal{L}, \mathcal{L}) \rightarrow \mathrm{Ext}_G^*(j^*\mathcal{L}, j^*\mathcal{L}) \cong \mathrm{Ext}_G^*(j^!\mathcal{L}, j^!\mathcal{L}),$$

where we used $j^* \cong j^!$ for an open embedding j . It follows that this map, viewed as a left A -module, is given by

$$\mathrm{Ext}_G^*(\mathcal{L}, \mathcal{L}) \rightarrow \mathrm{Ext}_G^*(j_!j^!\mathcal{L}, \mathcal{L}).$$

We have a distinguished triangle

$$\rightarrow j_!j^!\mathcal{L} \rightarrow \mathcal{L} \rightarrow i_*i^*\mathcal{L} \xrightarrow{+1}.$$

It yields the following short exact sequence of graded A -modules:

$$0 \rightarrow \mathrm{coker} \varpi \langle 1 \rangle \rightarrow K \xrightarrow{f} A \xrightarrow{\varpi} A' \rightarrow \mathrm{coker} \varpi \rightarrow 0.$$

By the purity assumption $(\clubsuit)_2$, the sheaf $i_*i^*\mathcal{L}$ is a direct sum of $\{\mathbb{C}_\mu[k]\}_{\mu \sim \lambda, k \in \mathbb{Z}}$. It follows that K is a direct sum of grading shifts of $\{\tilde{K}_\mu\}_{\mu \sim \lambda}$. Therefore, the

second assertion modulo the surjectivity of ϖ follows. We have an exact sequence:

$$0 \longrightarrow \operatorname{coker} \varpi \langle 1 \rangle \rightarrow K \xrightarrow{f} A \longrightarrow \operatorname{Im} \varpi \rightarrow 0. \quad (4.7)$$

By Theorem 1.8 3), we have $\tilde{K}_\mu = P_\mu$ for $\mu \sim \lambda$. Regarding (4.7) as the first two terms of a projective resolution, we deduce

$$\operatorname{ext}_A^1(\operatorname{Im} \varpi, L_\gamma) = \{0\} \quad \text{for every } \gamma \not\sim \lambda, \quad \text{and} \quad (4.8)$$

$$\operatorname{ext}_A^i(\operatorname{coker} \varpi, L_\gamma) \langle -1 \rangle \cong \operatorname{ext}_A^{i+2}(\operatorname{Im} \varpi, L_\gamma) \quad \text{for every } i \in \mathbb{Z}_{\geq 0} \quad \text{and } \lambda \not\sim \gamma \in \Lambda. \quad (4.9)$$

In addition, we have $[A' : L_\mu]_A = 0$ for $\mu \sim \lambda$ since L_μ is the multiplicity space of IC_μ in \mathcal{L} and $j^! \operatorname{IC}_\mu = \{0\}$. Moreover, we have

$$[\operatorname{Im} \varpi : L_\mu]_A \leq [A' : L_\mu]_A = 0, \quad \text{and} \quad [\operatorname{coker} \varpi : L_\mu]_A \leq [A' : L_\mu]_A = 0$$

for each $\mu \sim \lambda$. This, together with (4.8) and the shape of K , implies that $\operatorname{Im} \varpi \cong A/(Ae_\lambda A)$. The surjectivity of ϖ is equivalent to $\operatorname{coker} \varpi = \{0\}$, and it is further equivalent to

$$\operatorname{ext}_A^2(\operatorname{Im} \varpi, L_\gamma) = \operatorname{ext}_A^2(A/(Ae_\lambda A), L_\gamma) = \{0\} \quad \text{for every } \gamma \not\sim \lambda$$

by (4.9). This follows from Corollary 4.2 as desired. \square

Corollary 4.4. *Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G -stable subvariety. Then, \mathfrak{Y} satisfies the conditions (\spadesuit) and (\clubsuit) if \mathfrak{X} does.*

Proof. The conditions (\spadesuit) and $(\clubsuit)_2$ become weaker by the restriction to any G -invariant locally closed subset.

Let \mathbb{O}_λ be a closed G -orbit of \mathfrak{X} . We set $\mathbb{O}_\lambda = \mathfrak{X} \setminus \mathfrak{Y}$. By Proposition 4.3, we have a natural short exact sequence as graded left $A_{(G, \mathfrak{X})}$ -modules

$$0 \rightarrow K \rightarrow A_{(G, \mathfrak{X})} \rightarrow A_{(G, \mathfrak{Y})} \rightarrow 0. \quad (4.10)$$

This particularly shows the condition $(\clubsuit)_1$ for \mathfrak{X} implies that of \mathfrak{Y} when $\mathfrak{X} \setminus \mathfrak{Y}$ is a single G -orbit. Hence, we deduce the result by induction. \square

Remark 4.5. Thanks to Corollary 4.4, analogous results of Corollary 4.2 and Proposition 4.3 also hold for an arbitrary inclusion of closed G -stable subsets.

5 A proof of Shoji's conjecture for type B

In this section, we work with varieties over \mathbb{C} unless stated otherwise.

Let $G = Sp(2n, \mathbb{C})$ be a symplectic group, and let V_1 be its vector representation. We set $V_2 := \wedge^2 V_1$ and $\mathbb{V} := V_1 \oplus V_2$. We fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ so that $T \subset B$. Let $W := N_G(T)/T$. Let $X^*(T)$ be the character lattice of T , which we may identify with the cocharacter lattice via a W -invariant perfect pairing. We fix a basis $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ of $X^*(T)$ so that the set R of coroots of G and the set R^+ of positive coroots with respect to B are presented as:

$$R := \{\pm \epsilon_i \pm \epsilon_j\}_{i < j} \cup \{\pm \epsilon_i\}_{i=1}^n \supset \{\epsilon_i \pm \epsilon_j\}_{i < j} \cup \{\epsilon_i\}_{i=1}^n =: R^+.$$

For each $\beta \in X^*(T)$, let $\mathbb{V}[\beta]$ be the weight β -part of \mathbb{V} . Note that we have $\dim \mathbb{V}[\beta] \leq 1$ if $\beta \neq 0$. We set $\mathbb{V}^+ := \bigoplus_{\beta \in R^+} \mathbb{V}[\beta] \subset \mathbb{V}$, that is a B -submodule. We consider a G -equivariant vector bundle $F := G \times^B \mathbb{V}^+$ and a composition map

$$\mu : F \hookrightarrow G \times^B \mathbb{V} \cong G/B \times \mathbb{V} \xrightarrow{\text{pr}_2} \mathbb{V},$$

which is again G -equivariant. We denote the image of μ by \mathfrak{N} and denote the resulting map $F \rightarrow \mathfrak{N}$ again by μ .

For each $x \in \mathfrak{N}(\mathbb{C})$, the composition map $\mu^{-1}(x) \hookrightarrow F \rightarrow G/B$ is injective. It induces a map

$$\iota_x : H_\bullet(\mu^{-1}(x), \mathbb{C}) \longrightarrow H_\bullet(G/B, \mathbb{C})$$

between the Borel-Moore homologies (see [CG97] §2 for example).

We put $\mathbb{G} = G \times (\mathbb{C}^\times)^2$. An element $a = (s, q_1, q_2) \in \mathbb{G}$ acts on $v_1 \oplus v_2 \in \mathbb{V}$ as $a.v = q_1 s v_1 \oplus q_2 s v_2$.

Theorem 5.1 ([K09, K11] and Lusztig-Spaltenstein [LS79]). *We have:*

1. *The variety \mathfrak{N} has finitely many G -orbits, and a G -orbit is also a \mathbb{G} -orbit;*
2. *For each $x \in \mathfrak{N}(\mathbb{C})$, the groups $\text{Stab}_G(x)$ and $\text{Stab}_{\mathbb{G}}(x)$ are connected;*
3. *The map μ is strictly semi-small, and hence $\mu_* \mathbb{C}[\dim F]$ is a direct sum of G -equivariant simple perverse sheaves;*
4. *The sheaf $\mathcal{L} := \mu_* \mathbb{C}[\dim F]$ contains all G -equivariant simple perverse sheaves on \mathfrak{N} as its direct summands;*
5. *We have*

$$A := \text{Ext}_{D_G^b(\mathfrak{N})}^\bullet(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{t}],$$

where $\mathbb{C}W$ is a group algebra of W sitting in degree 0 and $\mathbb{C}[\mathfrak{t}]$ is a polynomial algebra generated by \mathfrak{t}^ in degree 2;*

6. *The odd-part of the Borel-Moore homology $H_{\text{odd}}(\mu^{-1}(x), \mathbb{C})$ vanishes;*
7. *Let $d_x = 2 \dim \mu^{-1}(x)$. Then, $\iota_x H_{d_x}(\mu^{-1}(x), \mathbb{C})$ is an irreducible W -submodule of $H_\bullet(G/B, \mathbb{C})$ that we denote by \mathbb{L}_x ;*
8. *Let $H_{-\bullet}(G/B, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}^*]$ be the harmonic polynomial realization ([CG97] §6 with the convention $\deg \mathfrak{t} = -2$; here \mathfrak{t}^* acts on $H_{-\bullet}(G/B, \mathbb{C})$ via derivatives with degree two). Then, the module \mathbb{L}_x is the unique maximal degree realization (of an irreducible W -module);*
9. *There exists a reflection subgroup $W_x \subset W$ and a polynomial $p_x \in \mathbb{L}_x$ so that $\mathbb{C}p_x$ is the maximal degree realization of a one-dimensional representation of W_x inside $\mathbb{C}[\mathfrak{t}^*]$.*

Proof. The first two assertions for G are in [K09] 1.14. The coincidence of G -orbits and \mathbb{G} -orbits follows by the form of a representative of each orbit *loc. cit.* 1.13. The second assertion for \mathbb{G} can be deduced from *loc. cit.* 4.10 and its proof. The third assertion follows by [K09] 1.2 and [K11] 3.4, the fourth is [K09] 8.3, the fifth is [K09] 8.1, and the sixth is [K09] 6.2. The seventh assertion follows from [K11] 7.6, 10.7 and [CG97] 6.5.2 (or rather its proof). The eighth and the ninth assertions follow from [LS79] (cf. [K11] 10.5). Here if we denote

the bipartition parametrizing the G -orbit $G.x$ by (μ, ν) by [K11] 5.1, then we have $p_x = D(\mu, \nu)$ in *loc. cit.* 10.4. In addition, W_x is the product of smaller Weyl groups of type BC whose sizes are the entries of ${}^t\mu$ and ${}^t\nu$. \square

Since $\mathfrak{t}^* \subset A$ in Theorem 5.1 5) is the space of (virtual) hyperplane sections (cf. [K09] §8), we deduce that each ι_x is an A -module map.

The G -variety \mathfrak{N} is a specialization of a flat \mathbb{Z} -scheme $\mathfrak{N}_{\mathbb{Z}}$ defined by the same defining equations as \mathfrak{N} (namely G -invariant polynomials on \mathbb{V} with integer coefficients and constant terms zero; cf. [K11] §2) to \mathbb{C} . Hence, for an algebraic closure \mathbb{k} of a field with p -element, we have a variety $\mathfrak{N}_{\mathbb{k}} (:= \mathfrak{N}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{k})$ over \mathbb{k} with an action of $G_{\mathbb{k}} (:= Sp(2n, \mathbb{k}))$.

Proposition 5.2. *For $p \gg 0$, the conditions (\spadesuit) in Condition 1.1 and (\clubsuit) in Condition 1.4 holds for $\mathfrak{N}_{\mathbb{k}}$. In addition, we have $A \cong A_{(G_{\mathbb{k}}, \mathfrak{N}_{\mathbb{k}})}$ in the notation of section one.*

Moreover, the algebra A and its standard/dual standard modules $K_{\lambda}/\tilde{K}_{\lambda}$ defined from \mathfrak{N} (over \mathbb{C}) satisfies the conclusions of Theorem 1.8, Proposition 3.3, and Corollaries 3.11 and 3.12.

Proof. As in [BBD82] §6 and [BL94], we transplant the notion of (equivariant) derived category of mixed constructible/perverse sheaves from the sufficiently large characteristic case to the characteristic zero case by (the scalar extension of) [BBD82] 6.1.9 (note that our sheaves have finite monodromy by its G -equivariance and $\#(G \backslash \mathfrak{N}) < \infty$, and such constructible $\overline{\mathbb{Q}}_{\ell}$ -sheaves are in common over \mathbb{k} and over \mathbb{C} ; cf. [BBD82] 6.1.2A”).

The condition that (\spadesuit) and (\clubsuit) holds for sufficiently large characteristic is equivalent to that to be hold in characteristic zero. In addition, the latter part of the assertion follows by the identification of their Hom-spaces. Therefore, it suffices to verify the conditions (\spadesuit) and (\clubsuit) in characteristic zero.

Theorem 5.1 1) and 2) asserts (\spadesuit) , and Theorem 5.1 3) and 4) imply that the algebra A (in Theorem 5.1 5)) is the one we described in section one. The existence of an α -partition (cf. [DLP88] 1.3) of the variety $F \times_{\mathfrak{N}} F$ into G -equivariant vector bundles on G/B (corresponding to equi-dimensional irreducible components; all of them are $\dim F = 2 \dim G/B$ dimensional) can be read off from [K09] 1.5 and 2.3. It follows that the G -equivariant Borel-Moore homology

$$H_{\bullet}^G(F \times_{\mathfrak{N}} F) := H^{-\bullet}(BG, p_* p^! \mathbb{C})$$

(where $p : EG \times^G (F \times_{\mathfrak{N}} F) \rightarrow BG$) is generated by its top term by the $\mathbb{C}[t]$ -action (arising from $H_G^{\bullet}(G/B, \mathbb{C})$ with the convention $\deg \mathfrak{t}^* = -2$) by [CG97] §5. By the isomorphism $A \cong H_{\bullet}^G(F \times_{\mathfrak{N}} F)$ as an algebra (cf. [CG97] §8), we conclude that A is generated by \mathfrak{t}^* and $\text{Hom}_G(\mathcal{L}, \mathcal{L})$. Here the both of \mathfrak{t}^* and $\text{Hom}_G(\mathcal{L}, \mathcal{L})$ are weight zero (actually we count this on $\mathfrak{N}_{\mathbb{k}}$ for $p \gg 0$ to verify it on \mathfrak{N}), and hence $(\clubsuit)_1$.

We show $(\clubsuit)_2$ by Theorem 1.6. Since (\spadesuit) and $(\clubsuit)_1$ also holds for G replaced with \mathbb{G} , we can use the \mathbb{G} -action instead of the G -action. By Theorem 5.1 6), the condition $a)'$ of Theorem 1.6 follows.

We verify Theorem 1.6 $b)$. Let $\mu \prec \lambda \in \Lambda$ (we borrow the notation from section one as we have (\spadesuit)).

Claim C. *There exists a semisimple element $A_{\mu} \in (\text{Lie } \mathbb{G})(\mathbb{R})$ which gives $a_{\mu} := \exp A_{\mu} \in \mathbb{G}$ so that $\mathbb{O}_{\mu}^{a_{\mu}} \subset \mathbb{V}^{a_{\mu}}$ is open dense.*

Proof. By [K09] §1.3, each x_μ is presented by a signed partition $\mathbf{J} = \{J_1, J_2, \dots\}$ dividing $[1, n]$ and a foot function $\delta_1 : [-n, n] \setminus \{0\} \rightarrow \{0, 1\}$ corresponding to a strict normal form *loc. cit.* 1.14. Fix $A_\mu := (S, r_1, r_2) \in (\text{Lie } T)(\mathbb{R}) \oplus \mathbb{R}^{\oplus 2}$ so that $a_\mu x_\mu = x_\mu$, $r_1, r_2 > 0$, $2r_1 \notin \mathbb{Z}r_2$, and

$$\{\pm \epsilon_i(S)\}_{i \in J_k^+} \cap \left\{ (\{\pm \epsilon_i(S)\}_{i \in J_{k'}^+} + \mathbb{Z}r_2) \cup \frac{1}{2}\mathbb{Z}r_2 \right\} = \emptyset$$

for each k, k' so that $\delta_1(J_k) = \{0\}$. Let $\mathbf{J}^0 := \{J_k \mid \delta_1(J_k) = \{0\}\}$. Then, the centralizer G^μ of A_μ in G is naturally contained in $Sp(2n_0) \times \prod_{J_k \in \mathbf{J}^0} GL(|J_k|)$, where $n_0 := n - \sum_{J_k \in \mathbf{J}^0} |J_k|$. Moreover, \mathfrak{N}^{a_μ} is contained in the product of nilpotent cones of $GL(|J_k|)$ and smaller \mathfrak{N} for $Sp(2n_0)$ (say \mathfrak{N}_{n_0}), and hence we have a product decomposition of the situation. Here x_μ is decomposed into the product of regular nilpotent elements of $GL(|J_k|)$ and an element $x'_\mu \in \mathfrak{N}_{n_0}$ (cf. *loc. cit.* 1.11). Thus, each $GL(|J_k|)$ -part defines a dense open subset.

Hence, we only need to consider the case of strict marked partition with $\delta_1(J_k) \neq \{0\}$ for all J_k (i.e. the case $n = n_0$). Decompose $x_\mu = x_\mu^1 \oplus x_\mu^2 \in V_1 \oplus V_2$. The condition $2r_1 \notin \mathbb{Z}r_2$ guarantees that $G^\mu x_\mu^2 \subset V_2^{a_\mu}$ is open dense. Now $G^\mu x_\mu$ gives an open dense subset of a rank $\dim V_1^{a_\mu}$ vector bundle over $G^\mu x_\mu^2$, and hence $G^\mu x_\mu \subset \mathfrak{N}^{a_\mu}$ is open dense. \square

We return to the proof of Proposition 5.2. Thanks to Claim C, we have $\mathbb{O}_\lambda^{a_\mu} = \emptyset$. It follows that A_μ is \mathbb{G} -conjugate to an element of $\text{Lie Stab}_\mathbb{G}(x_\mu)$, but is not \mathbb{G} -conjugate to an element of $\text{Lie Stab}_\mathbb{G}(x_\lambda)$. Passing to the adjoint quotients, we have maps

$$\text{Lie Stab}_\mathbb{G}(x_\mu) // \text{Stab}_\mathbb{G}(x_\mu) \xrightarrow{\vartheta_\mu} \text{Lie } \mathbb{G} // \mathbb{G} \xleftarrow{\vartheta_\lambda} \text{Lie Stab}_\mathbb{G}(x_\lambda) // \text{Stab}_\mathbb{G}(x_\lambda).$$

The point $A_\mu // \mathbb{G}$ lies in the image of ϑ_μ , but not lie on the image of ϑ_λ . In particular, evaluation at A_μ defines proper ideals of $H_\mathbb{G}^\bullet(\text{pt})$ and $H_\mathbb{G}^\bullet(\mathbb{O}_\mu)$, but it does not define a proper ideal of $H_\mathbb{G}^\bullet(\mathbb{O}_\lambda)$. Therefore, the natural map $\psi_\mu : H_\mathbb{G}^\bullet(\text{pt}) \rightarrow H_\mathbb{G}^\bullet(\mathbb{O}_\mu)$ does not factor through $\text{Im } \psi_\lambda$, which verifies the condition of Theorem 1.6 b) as required. \square

Corollary 5.3. *For each $x \in \mathfrak{N}(\mathbb{C})$, the variety $\mu^{-1}(x)$ has a pure homology.*
 \square

Thanks to Proposition 5.2, we can apply the machinery and notation of section one. In particular, we have $\underline{\Lambda} = G \backslash \mathfrak{N}$ and its closure relation \prec . We set $A_n := A$ and replace x with $\lambda \in \Lambda$ such that x is G -conjugate to $x_\lambda \in \mathbb{O}_\lambda$ in the below.

Lemma 5.4. *Under the above setting, the followings hold:*

1. *We have $C_\lambda = \{1\}$ for every $\lambda \in \Lambda$. In particular, we have $\Lambda = \underline{\Lambda}$;*
2. *Each of the graded simple self-dual A -module L_λ is concentrated in degree zero;*
3. *For each $\lambda \in \Lambda$, the A -module L_λ can be identified with the irreducible W -module $\mathbb{L}_\lambda \langle d_\lambda \rangle$ through the embedding $\mathbb{C}W = A^0 \subset A$.*

Proof. The first assertion follows by Theorem 5.1 2). The second assertion follows by Theorem 5.1 3) and the fact that all representation of W is real. The third assertion follows by the presentation of A in Theorem 5.1 5), and conventions in Theorem 5.1 7), 8). \square

Let triv and sgn be the trivial and sign representations of W , respectively. We define Ssgn to be the unique one-dimensional representation of W so that $\text{Ssgn} \not\cong \text{sgn}$ as W -representations and $\text{Ssgn} \cong \text{sgn}$ as \mathfrak{S}_n -representations.

Proposition 5.5 ([K15] 10.7, cf. sign twisted from [K09] 8.3). *Let $\lambda_0, \lambda_1 \in \Lambda$ be the labels corresponding to W -representations sgn and Ssgn , respectively. We have $\mathbb{O}_{\lambda_0} = \{0\}$, and \mathbb{O}_{λ_1} is the open dense G -orbit in $V_1 \subset \mathfrak{N}$ (here triv corresponds to the open dense G -orbit of \mathfrak{N}). \square*

For each $\lambda \in \Lambda$, we have

$$\begin{aligned} K_\lambda &= H^\bullet i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda] \cong \bigoplus_{i \in \mathbb{Z}} H^i i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda] \\ &= \bigoplus_{i \in \mathbb{Z}} H^i i_\lambda^! \mu_* \underline{\mathbb{C}}[\dim F + \dim \mathbb{O}_\lambda] \cong \bigoplus_{i \in \mathbb{Z}} H^i(\mu^{-1}(x_\lambda), \omega[-d_\lambda]), \end{aligned}$$

where ω is the dualizing complex of $\mu^{-1}(x_\lambda)$ and the last equation follows from the base change (and strict semi-smallness for the grading). This implies that $K_\lambda \cong H_{-\bullet}(\mu^{-1}(x_\lambda), \mathbb{C}) \langle d_\lambda \rangle$ as graded A -modules.

Recall that each K_λ is a graded A -module presented as:

$$K_\lambda = P_\lambda / \sum_{f \in \text{hom}_A(P_\mu, P_\lambda) > 0, \mu \preceq \lambda} \text{Im } f \quad (5.1)$$

by Theorem 1.8 3) and 4), and Theorem 5.1 2). The main result in this section is:

Theorem 5.6. *For each distinct $\lambda, \mu \in \Lambda$, we have*

$$\text{ext}_A^\bullet(K_\lambda, K_\mu^*) = \{0\}, \quad \text{and} \quad \langle K_\lambda, K_\mu^* \rangle = 0. \quad (5.2)$$

In addition, each ι_λ is an inclusion of A -modules.

Before proving Theorem 5.6, let us summarize its consequences. For each $\lambda \in \Lambda$, we define an ideal

$$I_\lambda := \{f \in \mathbb{C}[\mathfrak{t}] \mid fL_\lambda = \{0\}\}.$$

Corollary 5.7 (Shoji's conjecture [Sho04] 3.13). *For each $\lambda \in \Lambda$, the module $K_\lambda \langle -d_\lambda \rangle$ is identified with the span of derivatives of L_λ in the polynomial ring $\mathbb{C}[\mathfrak{t}^*]$. In addition, we have $K_\lambda^* \langle d_\lambda \rangle \cong \mathbb{C}[\mathfrak{t}]/I_\lambda$ as graded A -modules.*

Proof of Corollary 5.7 modulo Theorem 5.6. The module K_λ is spanned by L_λ by (5.1). Since $A \cong \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}} \mathbb{C}W$ and L_λ is W -stable, the image of ι_λ is exactly the span of derivatives of $L_\lambda = \iota_\lambda L_\lambda \langle -d_\lambda \rangle$ in the space of (harmonic) polynomials. Therefore, the latter half of Theorem 5.6 implies the former part of the assertion. The latter part of the assertion is a standard consequence of the former part of the assertion (see e.g. [Ems78] B.1). \square

For each $\lambda, \mu \in \Lambda$, we define $K_{\mu, \lambda}(t) \in \mathbb{Q}(t)$ by

$$[K_{\lambda}^*] = \sum_{\mu \in \Lambda} [\overline{K_{\lambda} : L_{\mu}}][L_{\mu}] = \sum_{\mu \in \Lambda} t^{-d_{\lambda}} K_{\mu, \lambda}(t^2)[L_{\mu}] \in K(A\text{-gmod}).$$

Corollary 5.8 (Numerical part of Shoji's conjecture [Sho04] 3.13). *For each $\lambda, \mu \in \Lambda$, $K_{\mu, \lambda}(t)$ is a polynomial with non-negative integer coefficients. In addition, $\{K_{\mu, \lambda}(t)\}_{\lambda, \mu \in \Lambda}$ are the modified Kostka polynomials of type B attached to the limit symbols.*

Remark 5.9. 1) Corollary 5.7, Theorem 1.8 1), 4), and Corollary 3.12 (cf. Proposition 5.2) implies that the graded W -character $t^{d_{\lambda}}[K_{\lambda}^*]$ (cf. Lemma 5.4 3)) of the ring $\mathbb{C}[\mathfrak{t}]/I_{\lambda}$ is computable from

$$K_{\mu, \lambda}(t) = \begin{cases} 0 & (\lambda \not\leq \mu) \\ t^{d_{\lambda}/2} & (\lambda = \mu) \end{cases}, \quad \text{and} \quad [P : L] = {}^{\mathfrak{t}}[K : L][\tilde{K} : K][K : L], \quad (5.3)$$

where $[P : L] = ([P_{\lambda} : L_{\mu}])_{\lambda, \mu \in \Lambda}$, $[K : L] = (t^{d_{\lambda}} K_{\mu, \lambda}(t^{-2}))_{\lambda, \mu \in \Lambda}$ are square matrices, and $[\tilde{K} : K]$ is a diagonal matrix in Corollary 3.12. In fact, solving the equation (5.3) determines $[K : L]$ and $[\tilde{K} : K]$ simultaneously from $[P : L]$ in a unique fashion (the *Lusztig-Shoji algorithm*). We have

$$[P_{\lambda} : L_{\mu}] = \text{gdim } \text{hom}_W(L_{\mu}, L_{\lambda} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{t}]),$$

that is commonly called the *fake degree* in the literature (with t replaced with $t^{1/2}$; cf. [K15] §2). This is the original form of Shoji's conjecture [Sho04] 3.13. 2) Corollary 5.8 is independently obtained by Shoji-Sorlin [SS12] (cf. Achar-Henderson [AH08] for a closely related result). Also, [SS12] contains another proofs of Corollaries 5.3 and 5.10.

Proof of Corollary 5.8 modulo Theorem 5.6. Each $K_{\mu, \lambda}(t)$ is a polynomial by Theorem 5.1 6) and $K_{\lambda}^i = H_{-i+d_{\lambda}}(\mu^{-1}(x_{\lambda})) = \{0\}$ for $i > d_{\lambda}$. It has non-negative integer coefficients as it counts the graded dimension of a module.

By [K15] 2.17 and (5.2), we deduce that $t^{d_{\lambda}/2} K_{\mu, \lambda}(t^{-1})$ is the Kostka polynomial attached to the preorder \prec on $\Lambda = \text{lrr } W$ in the sense of Shoji [Sho04] 1.3. Hence, in order to identify $\{K_{\mu, \lambda}(t)\}_{\lambda, \mu \in \Lambda}$ with the Kostka polynomials attached to the limit symbol, we need to identify the preorder relations on $\text{lrr } W$ arising from the exotic Springer correspondence and the limit symbols, and to identify the dimensions of the exotic Springer fibers and the a -function arising from the latter context. These identifications follow from Achar-Henderson [AH08] 6.3 with [Sho04] 1.4, and the fact that the b -function in [AH08] §5, our $d_{\lambda}/2$, and $n(\beta) = a(\Lambda(\beta))$ in [Sho04] §3 are in common. \square

Proof of Theorem 5.6. We apply Corollary 3.11 to deduce the first two equation.

We prove the remaining assertion. Let $M_{\lambda} := {}_{\iota_{\lambda}} K_{\lambda} \langle -d_{\lambda} \rangle$. Since K_{λ} has simple head as a graded A -module, the A -module M_{λ} is obtained by the A -module saturation of L_{λ} . By (5.1) and Theorem 1.8 1), the injectivity of ι_{λ} is equivalent to

$$\text{ext}_A^1(M_{\lambda}, L_{\mu}) = \{0\} \quad \text{for every } \lambda \prec \mu. \quad (5.4)$$

If $W_{\lambda} = W$ (recall that $W_{\lambda} = W_{x_{\lambda}}$ in Theorem 5.1 9) by convention), then we have $L_{\lambda} \cong \text{sgn}$ or Ssgn by [K15] Fact 4.1. We prove the assertion (5.4) in the

both cases. By Proposition 5.5, these cases correspond to **a)** $x_\lambda = 0$, and **b)** $\mathbb{O}_\lambda = (V_1 \setminus \{0\}) \oplus \{0\} \subset \mathbb{V}$, respectively. For the case **a)**, the assertion $M_\lambda = K_\lambda \langle -d_\lambda \rangle$ is standard since the both sides are $(\mathbb{C}[t]/\langle \mathbb{C}[t]_+^W \rangle)^*$ and $H_\bullet(G/B)$, respectively (cf. [CG97] §6.4). For the case **b)**, the $(G$ -equivariant) composition map $F \rightarrow \mathfrak{N} \rightarrow V_1$ is surjective, and is regular along x_λ . It follows that $\mu^{-1}(x_\lambda)$ is a $(d_\lambda/2)$ -dimensional smooth projective variety. In particular, we have the Poincaré duality $K_\lambda^* \cong K_\lambda \langle -d_\lambda \rangle$, which intertwines the action of the hyperplane sections. Since $\dim \mathbb{L}_\lambda = 1$, the module K_λ has simple head as a $\mathbb{C}[t]$ -module. Therefore, we conclude that $K_\lambda \langle -d_\lambda \rangle$ must have simple socle of degree 0. By construction, we also have $M_\lambda^0 \neq \{0\}$. This forces $K_\lambda \langle -d_\lambda \rangle \cong M_\lambda$.

In the below, we examine the case $W_\lambda \neq W$. If we have $\lambda \prec \mu$, then $d_\lambda > d_\mu$ (see Theorem 5.1 3)) and Theorem 5.1 9) implies $\mathbb{C}p_\lambda \not\subset L_\mu$ as W_λ -modules.

Let $A_\lambda := \mathbb{C}W_\lambda \rtimes \mathbb{C}[t] \subset A$. We have $A_\lambda \neq A$ by $W_\lambda \neq W$. We define M_λ^\downarrow as the A_λ -submodule of M_λ generated by $\mathbb{C}p_\lambda$. We have

$$\mathbb{C}WM_\lambda^\downarrow = M_\lambda \quad (5.5)$$

by $\mathbb{C}WA_\lambda = A$. Since the both of $\mathbb{C}W$ and $\mathbb{C}W_\lambda$ are semi-simple algebras, a non-trivial element of $\text{ext}_A^1(M_\lambda, L_\mu)$ induces a non-trivial extension as $\mathbb{C}[t]$ -modules, and hence that as A_λ -modules. In particular, the non-vanishing of the LHS of (5.4) implies

$$\text{ext}_{A_\lambda}^1(M_\lambda^\downarrow, L_\mu) \neq \{0\} \quad \text{for some } \lambda \prec \mu$$

from (5.5). If we decompose $W_\lambda \cong W_{n_1} \times W_{n_2} \times \cdots \times W_{n_m}$ by the smaller type BC Weyl groups, then we have $A_\lambda \cong \boxtimes_{i=1}^m A_{n_i}$ with $\sum_{i=1}^m n_i = n$. For each $1 \leq i \leq m$, we have a standard module K_{λ_i} of A_{n_i} corresponding to $L_{\lambda_i} \cong \text{sgn}$ or Ssgn (as W_{n_i} -modules) so that $\boxtimes_{i=1}^m K_{\lambda_i}$ is generated by a W_λ -module isomorphic to $\mathbb{C}p_\lambda$.

By the above argument, we deduce $M_\lambda^\downarrow \cong \boxtimes_{i=1}^m K_{\lambda_i}$ as a graded A_λ -module (up to grading shifts). By Proposition 3.3 (cf. Corollary 2.10), each direct summand of the minimal projective resolution of M_λ^\downarrow as a graded A_λ -module is of the form $\boxtimes_{i=1}^m P_{\gamma_i}$ with $\gamma_i \preceq \lambda_i$ for every $i = 1, \dots, m$ up to a grading shift. Since the head of P_{γ_i} is L_{γ_i} as a $\mathbb{C}W_{n_i}$ -module, we conclude that

$$\text{hom}_{W_\lambda}(\boxtimes_{i=1}^m L_{\gamma_i}, L_\mu) \neq \{0\} \quad (5.6)$$

for some $\lambda \prec \mu$ if (5.4) fails. By Theorem 5.1 9), the W_λ -module $\boxtimes_{i=1}^m L_{\gamma_i}$ is realized inside $\mathbb{C}[t^*]$ only at degree

$$-\sum_{i=1}^m d_{\gamma_i} \leq -\sum_{i=1}^m d_{\lambda_i} = -d_\lambda < -d_\mu.$$

This shows that (5.6) cannot happen. Therefore, we conclude $\text{ext}_A^1(M_\lambda, L_\mu) = \{0\}$ also in the case $W_\lambda \neq W$. This proceeds the induction and finishes the proof. \square

Corollary 5.10 (Achar-Henderson [AH08] Conjecture 6.4 (1)). *For each $\lambda, \mu \in \Lambda$, we have $[K_\lambda : L_\mu] = t^k Q(t^4)$ for some $k \geq 0$ and $Q \in \mathbb{N}[t]$.*

Proof. By [K15] Fact 4.1 (1) and (6), if L_λ corresponds to a bi-partition $(\lambda^{(0)}, \lambda^{(1)})$, then each $L_\mu \subset (t^*)^{\otimes l} \otimes_{\mathbb{C}} L_\lambda$ ($l \geq 0$) corresponds to a bi-partition $(\mu^{(0)}, \mu^{(1)})$

with $|\mu^{(0)}| - |\lambda^{(0)}| \equiv l \pmod{2}$. Since $P_\lambda \cong \mathbb{C}[t] \otimes L_\lambda$ as graded W -modules (cf. [K15] §1), we deduce $[P_\lambda : L_\mu] = t^{k'} Q'(t^4)$ for some $k' \geq 0$ and $Q' \in \mathbb{N}[[t]]$. As K_λ is a quotient of P_λ , we conclude the result. \square

Appendix A: Coinvariants and the Lieb-McGuire systems

We work in the setting of section five. We set $\alpha_i^\vee := \epsilon_i - \epsilon_{i+1}$ for each $1 \leq i \leq n$ (with $\epsilon_{n+1} := 0$). Let $s_1, \dots, s_n \in W$ be the corresponding simple reflections. We fix two parameters $m, r \in \mathbb{R}$. Let $\mathcal{H}_{r,m}$ be the algebra that contains the group ring $\mathbb{C}W$ and the polynomial ring $\mathbb{C}[t] = \mathbb{C}[\epsilon_1, \dots, \epsilon_n]$ with the following properties:

1. $\mathcal{H}_{r,m} \cong \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{C}[t]$ as vector spaces;
2. For each $1 \leq i \leq n$ and each $f \in \mathbb{C}[t]$, we have

$$s_i \cdot f - s_i f \cdot s_i = \begin{cases} r \frac{f - s_i f}{\alpha_i^\vee} & (i \neq n) \\ mr \frac{f - s_n f}{\alpha_n^\vee} & (i = n) \end{cases},$$

where $f \mapsto {}^w f$ ($w \in W$) is the natural w -action on $\mathbb{C}[t]$.

Let \mathbf{e} be the idempotent of $\mathbb{C}W$ corresponding to the trivial representation. We have an isomorphism $\mathbb{C}[t] \ni f \mapsto f\mathbf{e} \in \mathcal{H}_{r,m}\mathbf{e}$, by which we regard $\mathbb{C}[t]$ as a representation of $\mathcal{H}_{r,m}$.

We define an anti-isomorphism $\dagger : \mathcal{H}_{r,m} \rightarrow \mathcal{H}_{-r,m}$ as:

$$s_i^\dagger := s_i, \text{ and } \epsilon_j^\dagger = -\epsilon_j \text{ for every } 1 \leq i, j \leq n.$$

The center of $\mathcal{H}_{r,m}$ is isomorphic to $\mathbb{C}[t]^W$ via the natural inclusion (see [Lus89]), and hence each W -orbit $W\gamma$ (that we may simply denote by γ) of \mathfrak{t} defines a central character $\mathbb{C}[t]^W \rightarrow \mathbb{C} (=:\mathbb{C}_\gamma)$. Let $\text{Irr}_\gamma \mathcal{H}_{r,m}$ be the set of isomorphism classes of irreducible $\mathcal{H}_{r,m}$ -modules with a central character γ .

Theorem A.1 (Heckman-Opdam [HO97]). *Let \mathcal{R} be the space of C^∞ -functions with respect to ξ_1, \dots, ξ_n for which ϵ_i acts by $\frac{\partial}{\partial \xi_i}$. We set $\mathbb{C}[\mathfrak{t}^*] := \mathbb{C}[\xi_1, \dots, \xi_n] \subset \mathcal{R}$.*

1. *There exists a $\mathcal{H}_{-r,m}$ -action on \mathcal{R} so that the pairing*

$$\mathbb{C}[\mathfrak{t}] \times \mathcal{R} \ni (P, f) \mapsto (Pf)(0) \in \mathbb{C} \tag{A.7}$$

interchanges the $\mathcal{H}_{r,m}$ -module structures on $\mathbb{C}[\mathfrak{t}]$ with the $\mathcal{H}_{-r,m}$ -module structures of $\mathbb{C}[\mathfrak{t}^]$ and \mathcal{R} as:*

$$(TP, f) = (P, T^\dagger f) \text{ for every } T \in \mathcal{H}_{r,m}, P \in \mathbb{C}[\mathfrak{t}], \text{ and } f \in \mathcal{R};$$

2. *For each $\gamma \in \mathfrak{t}$, there exists a unique non-zero function $\phi_\gamma \in \mathcal{R}$ up to scalar such that:*

$$\mathbb{C}[\mathfrak{t}]^W \phi_\gamma \cong \mathbb{C}_\gamma \text{ as } \mathbb{C}[\mathfrak{t}]^W\text{-modules; and } W \text{ acts on } \mathbb{C}\phi_\gamma \text{ by triv;}$$

3. *If we set $M_{\gamma,m} := \mathcal{H}_{-r,m}\phi_\gamma$, then it is irreducible as a $\mathcal{H}_{-r,m}$ -module.*

The algebra $\mathcal{H}_{r,m}$ specializes to A (in section five) by setting $r = 0$.

We consider the following condition (\star) on $a = (s, \vec{q}) \in \mathbb{G} = G \times (\mathbb{C}^\times)^2$:

$(\star)_0$ $\vec{q} = (q^m, q)$, where $q = e^r \in \mathbb{R}_{>1}$ and $m \in \mathbb{R}$;

$(\star)_1$ $s = \exp(\gamma)$ with $\gamma \in \mathfrak{t}(\mathbb{R})$.

Theorem A.2 (Standard modules, [K09] §7, §9). *Assume that $a \in \mathbb{G}$ satisfies (\star) . Let $v = v_1 \oplus v_2 \in \mathbb{V}$ such that $a.v = v$. We have a $\mathcal{H}_{-r,m}$ -module*

$$M_{(a,v)} := H_\bullet(\mu^{-1}(v)^a),$$

that is isomorphic to K_λ for some $\lambda \in \Lambda$ as W -modules.

Theorem A.3 (eDL correspondence, [K09] §10). *Assume that $a \in \mathbb{G}$ satisfies (\star) . Then, we have a one-to-one correspondence*

$$\text{Irr}_\gamma \mathcal{H}_{-r,m} \ni L_{(a,v)} \longleftrightarrow v \in Z_G(s) \backslash \mathbb{V}^a.$$

Moreover, $L_{(a,v)}$ is a quotient of $M_{(a,v)}$ as a $\mathcal{H}_{-r,m}$ -module.

Lemma A.4. *Let $L \subset \mathbb{C}[\mathfrak{t}^*]$ be a homogeneous W -submodule isomorphic to L_λ for some $\lambda \in \Lambda$. Then, we have*

$$K_\lambda \langle -d_\lambda \rangle \subset AL \quad \text{inside} \quad \mathbb{C}[\mathfrak{t}^*].$$

Proof. By Theorem 5.1 9), $\mathbb{C}p_\lambda \subset K_\lambda \langle -d_\lambda \rangle \subset \mathbb{C}[\mathfrak{t}^*]$ is a one-dimensional representation of W_λ that is its maximal degree realization. Let $\mathbb{C}[\mathfrak{t}^*]^{W_\lambda}$ denote the W_λ -invariant part of $\mathbb{C}[\mathfrak{t}^*]$.

There exists $q \in L$ for which $\mathbb{C}W_\lambda q \cong \mathbb{C}W_\lambda p_\lambda$ as (one-dimensional) W_λ -modules. It follows that we have a factorization

$$q = p_\lambda \cdot r \quad \text{with} \quad r \in \mathbb{C}[\mathfrak{t}^*]^{W_\lambda}.$$

There exists a homogeneous element $Q^+ \in \mathbb{C}[\mathfrak{t}]$ such that $Q^+ q = 1 \in \mathbb{C}[\mathfrak{t}]$ by the non-degeneracy of the pairing (A.7) restricted to $\mathbb{C}[\mathfrak{t}^*] \subset \mathcal{R}$. Here we can rearrange Q^+ if necessary to assume that $\mathbb{C}Q^+$ is isomorphic to $\mathbb{C}p_\lambda$ as a W_λ -module. It follows that Q^+ admits a factorization $Q^+ = PQ$, where $P \in \mathbb{C}[\mathfrak{t}]$ is the minimal degree realization of $\mathbb{C}p_\lambda$ inside non-negatively graded ring $\mathbb{C}[\mathfrak{t}]$, and Q is W_λ -invariant. By the comparison of degrees, we conclude that $0 \neq Qq \in \mathbb{C}p_\lambda$, which implies the result. \square

Theorem A.5. *Let $\gamma \in \mathfrak{t}$. Then, there exists $\lambda \in \Lambda$ so that the vanishing order of $M_{m,\gamma}$ along 0 induces a graded W -module structure equal to $K_\lambda \langle -d_\lambda \rangle$.*

Remark A.6. In Theorem A.1, the function ϕ_γ gives rise to a unique (up to scalars) solution of the Lieb-McGuire system so that $\mathbb{C}[\mathfrak{h}]^W$ acts by γ along the set of regular points of \mathfrak{h} (see [HO97] for detail). In this sense, the vanishing order filtration of $M_{m,\gamma}$ measures the structure of the Taylor series of ϕ_γ .

Proof of Theorem A.5. For each $f \in \mathcal{R}$, let $\text{lt } f$ denote the maximal degree non-zero homogeneous component of the Taylor expansion of f along 0 (remember that our degree counting on \mathcal{R} is nonpositive). Then, the vanishing-order filtration of $M_{m,\gamma}$ is transformed into the graded structure of the module

$$\text{gr } M_{m,\gamma} := \{\text{lt } f \mid f \in M_{m,\gamma}\}.$$

Since $M_{m,\gamma}$ is $\mathbb{C}[t]$ -stable, so is the space $\text{gr } M_{m,\gamma}$. Here [HO97] formula (2.1) implies that $s_i \in W$ acts on $\mathbb{C}[t^*]$ by letting s_i acts by the natural homogeneous action and adds some lower order terms. It follows that $\text{gr } M_{m,\gamma} \subset \mathbb{C}[t^*]$ is an A -submodule. Here $M_{m,\gamma}$ is a $\mathcal{H}_{-r,m}$ -module that contains triv as a W -module. It implies that the irreducible module $M_{m,\gamma}$ is isomorphic to some standard module in the sense of Theorem A.2 (cf. [CK11] 1.20). Hence, Theorem A.3 asserts that $\text{gr } M_{m,\gamma} \cong K_\lambda$ as a W -module for some $\lambda \in \Lambda$ (note that Theorem 1.8 1) determines such λ uniquely only from the W -module structure of $\text{gr } M_{m,\gamma}$). By Theorem 5.1 8), we deduce that the image of the natural composition map

$$\varphi : L_\lambda \hookrightarrow \text{gr } M_{m,\gamma} \subset \mathbb{C}[t^*]$$

lands in degree $\leq -d_\lambda$. Since $\dim \text{Hom}_W(L_\lambda, M_{m,\gamma}) = 1$, we conclude that $\text{Im } \varphi$ is homogeneous. If $\text{Im } \varphi = L_\lambda$ (i.e. the degree is $-d_\lambda$ by Theorem 5.1 8)), then we have $\text{gr } M_{m,\gamma} = K_\lambda \langle -d_\lambda \rangle$ by the inclusion as A -modules and the comparison of dimensions.

If $\text{Im } \varphi$ is of degree $< -d_\lambda$, then Lemma A.4 implies that $\dim \text{Hom}_W(L_\lambda, M_{m,\gamma}) \geq 2$. Hence, this case cannot occur as desired. \square

Appendix B: Proof of Theorem 1.6

This appendix is entirely devoted to the proof of Theorem 1.6. We assume the setting of section one. In particular, we assume (\spadesuit) in Condition 1.1, $(\clubsuit)_1$ in Condition 1.4, and work over an algebraic closure of a finite field. For each $\lambda \in \Lambda$, we have a natural morphism $\psi_\lambda : H_G^\bullet(\text{pt}) \rightarrow H_G^\bullet(\mathbb{O}_\lambda)$ of graded algebras. Let us restate our theorem to be proved:

Theorem B.1 (Theorem 1.6). *The sheaf IC_λ (normalized as weight zero) is pointwise pure of weight zero for every $\lambda \in \Lambda$ if the following two conditions hold:*

- a) *The following spectral sequence is E_2 -degenerate for each $\lambda = (\underline{\lambda}, \xi) \in \Lambda$:*

$$E_2 = \bigoplus_{\mu=(\underline{\lambda}, \zeta) \in \Lambda} \text{Hom}_{C_\lambda}(\xi, H_{G_\lambda}^\bullet(\{x_\lambda\}) \otimes_{\mathbb{C}} \zeta) \boxtimes K_\mu \Rightarrow \tilde{K}_\lambda. \quad (1.6)$$

- b) *We have $\ker \psi_\lambda \not\subset \ker \psi_\gamma$ for every $\gamma \prec \lambda \in \Lambda$;*

Here the condition a) can be replaced by its variant:

- a)' *For each $\lambda, \mu \in \Lambda$, the stalk of IC_λ along \mathbb{O}_μ satisfies the parity vanishing.*

We first prove that a)' implies a).

We assume a)'. Since G_λ° is an affine algebraic group, we deduce that the odd degree part of $H_{G_\lambda^\circ}^\bullet(\text{pt})$ is zero. It follows that we cannot have a non-zero differential among distinct terms of the spectral sequence

$$E_2(\gamma) := H_{G_\lambda^\circ}^\bullet(\{x_\lambda\}) \otimes_{\mathbb{C}} i_\lambda^! \text{IC}_\gamma \Rightarrow H_{G_\lambda^\circ}^\bullet(\{x_\lambda\}, i_\lambda^! \text{IC}_\gamma)$$

for each $\gamma \in \Lambda$. Since $E_2 = \text{Hom}_{C_\lambda}(\xi, \bigoplus_{\gamma \in \Lambda} L_\gamma \boxtimes E_2(\gamma))$ as spectral sequences of vector spaces, we conclude a).

Now we assume a) and b) to deduce that IC_λ (normalized as weight zero) is pointwise pure of weight zero along \mathbb{O}_γ for each $\gamma \in \Lambda$.

Lemma B.2. *The conditions (\spadesuit) and $(\clubsuit)_1$ are stable under the restriction to a G -stable closed subvariety $\mathfrak{Y} \subset \mathfrak{X}$.*

Proof. The condition (\spadesuit) is clear. For the condition $(\clubsuit)_1$, let us name the closed embedding $i : \mathfrak{Y} \hookrightarrow \mathfrak{X}$. For every $\mathcal{E}, \mathcal{F} \in D_G^b(\mathfrak{Y})$, we have

$$\mathrm{Ext}_{D_G^b(\mathfrak{Y})}^\bullet(\mathcal{E}, \mathcal{F}) \cong \mathrm{Ext}_{D_G^b(\mathfrak{Y})}^\bullet(i^* i_* \mathcal{E}, \mathcal{F}) \cong \mathrm{Ext}_{D_G^b(\mathfrak{X})}^\bullet(i_* \mathcal{E}, i_* \mathcal{F})$$

by adjunction. Hence, if we consider the idempotent $e \in \mathrm{End}_G(\mathcal{L})$ so that $e\mathcal{L} = \bigoplus_{\lambda \in \Lambda, \mathbb{O}_\lambda \subset \mathfrak{Y}} L_\lambda \boxtimes \mathrm{IC}_\lambda$, then we have $A_{(G, \mathfrak{Y})} = eA_{(G, \mathfrak{X})}e$. It follows that the condition $(\clubsuit)_1$ for \mathfrak{X} implies that of \mathfrak{Y} as required. \square

We assume that IC_λ is not pointwise pure of weight zero along \mathbb{O}_γ to deduce contradiction (we have $\lambda \succ \gamma$ by the support condition). By rearranging the choice of λ and γ and replacing \mathfrak{X} with \mathbb{O}_λ if necessary, we can assume:

- c) The orbit \mathbb{O}_λ is open dense in \mathfrak{X} ;
- d) For every $\gamma \prec \mu \prec \lambda$, the sheaf IC_μ is pointwise pure of weight zero along \mathbb{O}_γ .

Theorem B.3 (Theorem 2.8 + Corollary 2.7). *For each $\delta \in \Lambda$, we have a complex of graded projective A -modules*

$$(Q(\mathbb{C}_\delta), d) : \cdots \rightarrow Q(\mathbb{C}_\delta)_{-2} \xrightarrow{d_{-2}} Q(\mathbb{C}_\delta)_{-1} \xrightarrow{d_{-1}} Q(\mathbb{C}_\delta)_0 \rightarrow 0$$

with the following properties:

1. We have $Q(\mathbb{C}_\delta)_a \cong \mathrm{Ext}_G^\bullet(\mathrm{gr}_a \mathbb{C}_\delta, \mathcal{L})$ as a graded A -module with pure weight $-a$;
2. We have an isomorphism of graded A -modules:

$$\tilde{K}_\delta \cong \mathrm{Ext}_G^\bullet(\mathbb{C}_\delta, \mathcal{L}) \cong H^\bullet(Q(\mathbb{C}_\delta), d);$$

3. For each $a < 0$, the projective A -module $Q(\mathbb{C}_\delta)_a$ does not contain grading shifts of P_ϵ with $\epsilon \sim \delta$ as its direct factor.

Proof. We explain the diffusion from Theorem 2.8. The complex $(Q(\mathbb{C}_\delta), d)$ vanishes in positive degree by Corollary 2.7 1). The last property follows from Corollary 2.7 2). \square

For each $\mu \in \Lambda$, we have

$$K_\gamma \supset L_\mu \boxtimes i_\gamma^! \mathrm{IC}_\mu.$$

By the purity of $H_{G_\gamma}^\bullet(\mathrm{pt})$ and the condition a), we deduce that $[H^i(Q(\mathbb{C}_\gamma), d) : L_\mu] \neq 0$ holds if and only if the vector space $i_\gamma^! \mathrm{IC}_\mu$ contains nonzero weight $(-i)$ -part. By the condition d) and the definition of K_γ , we have

$$[H^i(Q(\mathbb{C}_\gamma), d) : L_\mu] = 0 \quad \text{if } \mu \not\prec \lambda$$

for each $0 \neq i \in \mathbb{Z}$. In addition, there exists $i \neq 0$ so that $H^i(Q(\mathbb{C}_\gamma), d) \neq \{0\}$ since $i_\gamma^! \mathrm{IC}_\lambda$ is not pure of weight zero by assumption. Henceforth we fix $i \neq 0$

with $H^i(Q(\mathbb{C}_\gamma), d) \neq \{0\}$. By the condition $a)$, we deduce that $H^i(Q(\mathbb{C}_\gamma), d)$ is a torsion-free $H_{G_\gamma}^\bullet(\text{pt})$ -module.

The graded A -module \tilde{K}_λ admits a finite length projective resolution by Theorem B.3 1)–2) (cf. Proposition 2.9). In addition, every irreducible constituent of \tilde{K}_λ is a grading shift of L_μ with $\mu \sim \lambda$ by the condition $c)$. Moreover, Theorem B.3 3) implies that

$$\text{ext}_A^1(\tilde{K}_\lambda, L_\mu) = \{0\} \quad \text{for every } \mu \sim \lambda.$$

This shows that \tilde{K}_λ is an indecomposable projective cover of L_λ in the category of graded A -modules with its composition factors in $\{L_\mu \langle j \rangle\}_{\mu \sim \lambda, j \in \mathbb{Z}}$. Therefore, the graded A -module $H^i(Q(\mathbb{C}_\gamma), d)$ is a quotient of a direct sum of grading shifts of \tilde{K}_μ with $\mu \sim \lambda$. By the condition $a)$, the graded A -module \tilde{K}_μ with $\mu \sim \lambda$ is a torsion-free $H_{G_\lambda}^\bullet(\text{pt})$ -module, and we have $\text{End}_A(\tilde{K}_\mu) \cong H_{G_\lambda}^\bullet(\text{pt})$ as graded algebras. This induces a $H_{G_\lambda}^\bullet(\text{pt})$ -action on $H^i(Q(\mathbb{C}_\gamma), d)$, that is non-trivial (but not necessarily faithful).

We have a $H_G^\bullet(\text{pt})$ -action on $H^i(Q(\mathbb{C}_\gamma), d)$ defined through the natural map $H_G^\bullet(\text{pt}) \rightarrow A$ which lands on the center of A (see (1.2)). These three actions must be compatible through ψ_λ and ψ_γ since the $H_G^\bullet(\text{pt})$ -action represents the action of (some part of) the center of A . In particular, if $\xi \in H_G^\bullet(\text{pt})$ belongs to $\ker \psi_\lambda$, then the action of $\psi_\lambda(\xi)$ must be zero, and hence $\psi_\gamma(\xi)$ annihilates $H^i(Q(\mathbb{C}_\gamma), d)$. Since $\psi_\gamma(\xi)$ cannot have torsion (as seen above), we conclude $\psi_\gamma(\xi) = 0$. It follows that $\ker \psi_\lambda \subset \ker \psi_\gamma$. This violates the condition $b)$, and we deduce a contradiction. Therefore, we conclude that every IC_λ must be pointwise pure as required.

References

- [AH08] Pramod N. Achar, and Anthony Henderson, Orbit closures in the enhanced nilpotent cone. *Adv. Math.* 219 (2008), 27–62.
- [Ach11] Pramod N. Achar, Green functions via hyperbolic localization, *Documenta Math.* 16 (2011) 869–884.
- [Ach13] Pramod N. Achar, Kostka systems and exotic t-structures for reflection groups, In: *Recent developments in algebraic and combinatorial aspects of representation theory*, 1–21, *Contemp. Math.*, 602, Amer. Math. Soc., Providence, RI, 2013
- [Ari96] Susumu Ariki, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$, *J. Math. Kyoto Univ.* 36, no. 4 (1996), 789–808.
- [ABG] Sergey Arkhipov, Roman Bezrukavnikov, and Victor Ginzburg, Quantum groups, the loop Grassmannian, and the Springer resolution. *J. Amer. Math. Soc.* 17, No. 3, 595–678 (2004)
- [BB81] Alexander Beilinson, and Joseph Bernstein, Localisation des \mathfrak{g} -modules, *C. R. Acad. Sci. Paris* 292 (1981), 15–18
- [BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne, Faisceaux pervers, *Astérisque* 100 (1982).
- [BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel, Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* 9 (1996), no. 2, 473–527.
- [Bon12] Mikhail V. Bondarko, Weight structures and ‘weights’ on the hearts of t-structures. *Homology Homotopy Appl.* 14 (2012), no. 1, 239–E61.
- [BKM12] Jonathan Brundan, Alexander Kleshchev, Peter J. McNamara, Homological properties of finite type Khovanov-Lauda-Rouquier algebras, arXiv:1210.6900

- [BK81] Jean-Luc Brylinski, and Masaki Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems, *Invent. Math.* 64 (1981), 387–410.
- [BL94] Joseph Bernstein, and Varelly Lunts, *Equivariant sheaves and functors*, Lecture Note in Math. 1578, Springer-Verlag 1994.
- [CK11] Dan Ciubotaru, and Syu Kato, Tempered modules in exotic Deligne-Langlands correspondences, *Adv. Math.* 226 no.2 1538–1590 (2011)
- [CG97] Neil Chriss, and Victor Ginzburg, *Representation theory and complex geometry*. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp.
- [CPS88] Edward Cline, Brian Parshall, and Leonard Scott, Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* 391 (1988) 85–99.
- [Del81] Pierre Deligne, La conjecture de Weil. II., *Inst. Hautes Études Sci. Publ. Math.* No. 52 (1980), 137–252.
- [DLP88] Corrado De Concini, George Lusztig, and Claudio Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, *J. Amer. Math. Soc.* 1 (1988), 15–34
- [Don98] Stephen Donkin, *The q -Schur Algebra*, Cambridge Univ. Press, 1999. x+179 pp.
- [Ems78] Jacques Emsalem, Géométrie des points épais, *Bull. Soc. Math. France* **106** (1978) 399–416
- [Gin85] Victor Ginzburg, *Deligne-Langlands conjecture and representations of affine Hecke algebras*, preprint, Moscow 1985.
- [GM11] Jérémie Guillhot, and Vanessa Miemietz, Affine cellularity of affine Hecke algebras of rank two, *Math. Z.* 271 (2012), no. 1–2, 373–397.
- [HO97] Gert Heckmann and Eric Opdam, Yang’s system of particles and Hecke algebras, *Ann. of Math.* **145** no.1 139–173 (1997)
- [Jos97] Roy Joshua, Modules over convolution algebras from equivariant derived categories, *J. Algebra*, (1997), 203, 385–446.
- [K09] Syu Kato, An exotic Deligne-Langlands correspondence for symplectic groups, *Duke Math. J.* 148 (2009), 305–371.
Errata: <http://www.math.kyoto-u.ac.jp/~syuchan/errata6.html>
- [K11] Syu Kato, Deformations of nilpotent cones and Springer correspondences, *Amer. J. Math.* **133** no.2 (2011) 519–553
- [K15] Syu Kato, A homological study of Green polynomials, *Ann. Sci. l’École Norm. Sup.* 4 48 no. 5 (2015) 1035–1074.
- [K14] Syu Kato, Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras, *Duke Math. J.* 163 (2014) 619–663.
- [KL87] David Kazhdan, and George Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, *Invent. Math.* 87 (1987), no. 1, 153–215.
- [KX12] Steffen König, and Changchang Xi, Affine cellular algebras, *Adv. Math.* 229, (2012), 139–182.
- [KP82] Hanspeter Kraft, and Claudio Procesi, On the geometry of conjugacy classes in classical groupes. *Commentarii mathematici Helvetici* **57** (1982) 539–602
- [Li96] Huishi Li, Global dimension of graded local rings, *Comm. Algebra*, 24, (1996), 2399–2405
- [LS79] George Lusztig, and Nicolas Spaltenstein, Induced unipotent classes, *J. London Math. Soc.*, 19, (1979), 41–52.
- [Lus88] George Lusztig, Cuspidal local systems and graded Hecke algebras, I, *Publ. Math. IHÉS*, **67** (1988), 145–202.
- [Lus89] George Lusztig, Affine Hecke algebras and their graded version. *J. Amer. Math. Soc.* **2** (1989), no. 3, 599–635.
- [Lu90a] George Lusztig, Canonical bases arising from quantized enveloping algebras, *J. Amer. Math. Soc.* 3 (1990) 447–498.
- [MR01] John C. McConnell, and James C. Robson, *Noncommutative Noetherian rings*. Revised edition. *Graduate Studies in Math.*, 30. AMS, 2001. ISBN: 0-82-182169-5

- [MV87] Renato Mirollo, and Kari Vilonen, Bernstein-Gelfand-Gelfand reciprocity on perverse sheaves. *Ann. Sci. l'École Norm. Sup, Sér. 4*, 20 no. 3 (1987), 311–323
- [Nak06] Hiraku Nakajima, Quiver varieties and quantum affine algebras, *Sugaku Expositions*, 19 (2006), no. 1, 53–78.
- [NO82] Constantin Năstăsescu, and Fred van Oystaeyen, *Graded ring theory*. North-Holland Mathematical Library, 28. North-Holland, 1982. ix+340 pp.
- [OS09] Eric Opdam, and Maarten Solleveld, Homological algebra for affine Hecke algebras, *Adv. Math.* 220 (2009), 1549–1601.
- [Pau08] David Pauksztello, Compact corigid objects in triangulated categories and co-t-structures. *Cent. Eur. J. Math.*, 6(1) 25–42, 2008
- [Rid12] Laura Rider, Formality for the nilpotent cone and a derived Springer correspondence, *Adv. Math.* 235 (2013), 208–236.
- [Sch11a] Olaf Schnürer, Equivariant sheaves on flag varieties, *Math. Zeit.* 267 (2011), 27–80
- [Sch11b] Olaf Schnürer, Homotopy categories and idempotent completeness, weight structures and weight complex functors, *arXiv:1107.1227*.
- [SW11] Catharina Stroppel, and Ben Webster, Quiver Schur algebras and q -Fock space, *arXiv:arXiv:1110.1115*
- [Sho83] Toshiaki Shoji, On the Green polynomials of classical groups, *Invent. Math.* 74 (1983), 239–267.
- [Sho88] Toshiaki Shoji, Geometry of orbits and Springer correspondence, *Astérisque* 168 (1988), 61–140.
- [Sho04] Toshiaki Shoji, Green functions attached to limit symbols, Representation theory of algebraic groups and quantum groups, 443–467, *Adv. Stud. Pure Math.*, **40**, Math. Soc. Japan, Tokyo, 2004.
- [SS12] Toshiaki Shoji, and Karine Sorlin, Exotic symmetric space over a finite field, II, *Transform. Groups* 19 (2014), no. 3, 887–926.
- [Soe90] Wolfgang Soergel, Kategorie \mathcal{O} , perverse Garben, und Moduln der den Koinvarianten zur Weylgruppe, *J. Amer. Math. Soc.* 3, 421–445 (1990).
- [Soe00] Wolfgang Soergel, Langlands's philosophy and Koszul duality, pp 379–414 in: *Proceedings of NATO ASI 2000, Algebra-Representation Theory*, Kluwer 2001.
- [VV99] Michela Varagnolo, and Eric Vasserot, On the decomposition matrices of the quantized Schur algebra, *Duke Math. J.* 100, (1999) 267–297.
- [VV11a] Michela Varagnolo, and Eric Vasserot, Canonical bases and KLR-algebras, *J. reine angew. Math.* 659 67–100 (2011).
- [VV11b] Michela Varagnolo, and Eric Vasserot, Canonical bases and affine Hecke algebras of type B, *Invent. Math.* 185 no.3 593–693 (2011)
- [Web10] Ben Webster, Knot invariants and higher representation theory I: diagrammatic and geometric categorification of tensor products, *arXiv:1001.2020*
- [Zhe08] Hao Zheng, Categorification of integrable representations of quantum groups, *Acta Math. Sin. (Engl. Ser.)* 30 (2014), no. 6, 899–932